f–harmonic maps and applications to gradient Ricci solitons

Giona Veronelli

Université de Cergy-Pontoise
Laboratoire Analyse Géométrie et Modélisation

Séminaire de Géométrie et Analyse
Laboratoire Dieudonné - Université de Nice

joint work with Michele Rimoldi

02 february 2012

Introduction

$(M^m, \langle\, , \rangle_M)$ and $(N^n, \langle\, , \rangle_N)$ complete Riemannian manifolds (usually N compact, ${}^N\mathrm{Sect} \leq 0$), $\dim M = m \geq 2$, $\dim N = n$.
Let $f \in C^\infty(M)$.

An f–harmonic map $u : M \to N$ is a stationary point of the f–energy

$$E_f^\Omega(u) = \frac{1}{2}\int_\Omega |du|^2 e^{-f}\, dV_M, \qquad \Omega \subset M.$$

dV_M is the volume measure on M.

Note. If $f \equiv const. \Rightarrow$

$$f\text{-harmonicity} \Leftrightarrow \text{harmonicity}$$

So far

Some (all?) references for f-harmonic maps:

- introduced by A. Lichnerowicz [Symp.Math.1968] on compact manifolds, $\dim M = m \geq 3$;
- J. Eells and L. Lemaire [Bull.London'78]: existence in $m = 2$;
- N. Course [New York M.J.'04]: $f-$harmonic flow on surfaces;
- S. Ouakkas, R. Nasri, and M. Djaa [JP J.Geom.Topol.'10]: $f-$biharmonic maps;
- Y.-L. Ou [arXiv'11]: $f-$harmonic morphisms.

Problem

1. *Why are f-harmonic maps so poorly investigated?*
2. *Are f-harmonic maps interesting objects?*

f-harmonic maps on M are strictly related to harmonic maps on some $\tilde{M}$ related to M; e.g. for $m \geq 3$,

$$E_f^M(u) = \int_M |du|^2 e^{-f} dV_M = \int_{\tilde{M}} |du|^2 dV_{\tilde{M}} = E^{\tilde{M}}(u),$$

where $\tilde{M} = (M, e^{-\frac{2f}{m-2}} \langle \, , \, \rangle_M) \Rightarrow$ maybe not interesting in their own.

Special case: $N = \mathbb{R} \Rightarrow f$-harmonic function with Euler-Lagrange equation

$$0 = \Delta_f u = {}^M\Delta u - \langle \nabla f, \nabla u \rangle_M$$

Weighted manifolds

Δ_f: natural diffusion operator associated to the weighted manifold (or manifold with density) $(M^m, \langle\, ,\, \rangle_M, e^{-f} dV_M)$.

Δ_f is useful because of its good interaction with curvature conditions, but no canonical choice for the right concept of curvature on a weighted manifolds.

We are interested in the (∞–)Bakry–Émery Ricci tensor

$$ {}^M\mathrm{Ric}_f = {}^M\mathrm{Ric} + \mathrm{Hess}\, f. $$

Rmk. If $\tilde{M} = (M, e^{-\frac{2f}{m-2}} \langle\, ,\, \rangle_M) \Rightarrow {}^M\mathrm{Ric}_f \neq {}^{\tilde{M}}\mathrm{Ric}$

Ric_f strictly related to Ricci solitons ("self–similar" solutions to Hamilton's Ricci flow).

Definition. A *Ricci soliton* structure on M is the choice of a vector field X satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X \langle\, , \,\rangle = \lambda \langle\, , \,\rangle , \qquad \lambda \in \mathbb{R}.$$

A Ricci solitons can be $\begin{cases} \text{shrinking} & \text{if } \lambda > 0 \\ \text{steady} & \text{if } \lambda = 0 \\ \text{expanding} & \text{if } \lambda < 0 \end{cases}$

Special case: gradient Ricci soliton, i.e. $X = \nabla f$, $f \in C^\infty(M)$, and

$$^M\,\mathrm{Ric}_f = {}^M\,\mathrm{Ric} + \mathrm{Hess}\, f. = \lambda \langle\, , \,\rangle_M , \qquad \lambda \in \mathbb{R}.$$

Target: understanding the topology of weighted manifolds and gradient Ricci solitons

Geometric context

Remark.

- Compact Ricci solitons, $\lambda \leq 0 \Rightarrow f \equiv const$.
- When M is non–compact and $\lambda \leq 0$, few information is known.

Concerning the shrinking side, i.e. $\text{Ric}_f = \lambda \langle \, , \, \rangle$, $\lambda > 0$, the classical Myers–Bonnet theorem cannot be extended to Ric_f: e.g. the *Gaussian space*

$$(\mathbb{R}^m, \langle \, , \, \rangle_{can}, e^{-|x|^2/2} dV_{\mathbb{R}^m})$$

is a non–compact, complete weighted manifold with $\text{Ric}_f = 1 > 0$.

$\Rightarrow$ one needs additional assumptions (e.g. $|df|$ bounded, assumption on radial growth of f,...)

Nevertheless

- ${}^{M}\mathrm{Ric}_f \geq c^2 > 0 \;\Rightarrow$ finite fundamental group
  - [Fernández-López–García-Río Math. Ann.'08]: for compact M;
  - [Wylie PAMS'08]: for non–compact M.
- If M is compact, ${}^{M}\mathrm{Ric}_f \geq 0$ and ${}^{M}\mathrm{Ric}_f > 0$ at one point then $|\pi_1(M)| < \infty$ [Cheeger-Gromoll JDG'71, Lichnerowicz].
- γ geodesic on M. If $\int_\gamma {}^{M}\mathrm{Ric}_f(\dot\gamma(t), \dot\gamma(t)) = +\infty \;\Rightarrow\; |\pi_1(M)| < \infty$ [Pigola-Rigoli-Rimoldi-Setti AnnSNSPisa'10].

We are interested in the cases $\lambda = 0$ (steady) and $\lambda < 0$ (expanding): very few is known.

- M non–compact, ${}^{M}\mathrm{Ric}_f \equiv 0 \;\Rightarrow$ either connected at infinity or isometric to a Ricci flat cylinder [Munteanu-Wang CAG'11]

If $|f|$ is bounded $\Rightarrow$ standard comparison results (laplacian, volume) generalize to Ric_f [Wei-Wylie JDG'09] $\Rightarrow$

Theorem

M non–compact, ${}^M\mathrm{Ric}_f \geq 0$, and $|f|$ bounded. Then

1. *M either satisfies the loops to infinity property or has a double covering which splits. In particular if ${}^M\mathrm{Ric}_f > 0$ then M satisfies the loops to infinity property [Sormani Indiana'01, W-W];*
2. *if $D \subset M$ is a compact with $\pi_1(\partial D) = \mathbf{1}$, $\Rightarrow$ $\pi_1(D)$ can only contain elements of order 2 [Sormani].*
3. *$b_1(M) \leq m$ [Milnor JDG'68, W-W].*

Standard techniques: use the properties of Δ_f and f-harmonic function to study weighted manifold and gradient Ricci solitons.

In this talk: use f-harmonic maps.

Harmonic maps

In the mid 1960's, Eells and Sampson introduced the notion of harmonic map.

Topological relevance (Hartman, Lemaire, Hamilton,...)**:**

- Characterize homotopy of a map $f : M \to N$
- Detect the topology of the involved manifolds

Typical approach: M (resp. N) is given.

- Choose suitable manifold N (resp. M) and map $u : M \to N$
- Prove $\exists\ v : M \to N$ harmonic, homotopic to u
- Harmonicity gives v is constant
- Then u homotopic to a constant $\Rightarrow$ restrictions on $u \Rightarrow$ on M

Examples:

- [Preismann] N compact, ${}^N\mathrm{Sect} < 0$. Then $\mathbb{Z}^2 \not\subset \pi_1(N)$.
- [Lohkamp] There is no metric $\langle , \rangle$ on $\mathbb{R}^m$ such that (a) $\langle , \rangle = can_{\mathbb{R}^m}$ on $\mathbb{R}^m \setminus \mathbb{B}_1$ and (b) $\mathrm{Ric} \geq 0$ in $\mathbb{B}_1$ and $\mathrm{Ric}(x_0) > 0$ for some $x_0 \in \mathbb{B}_1$.

When M is non–compact:

Theorem (Schoen-Yau, CommMathHelv'76)

M complete, ${}^M\mathrm{Ric} \geq 0$, and N compact, ${}^N\mathrm{Sect} \leq 0$. Let $u : M \to N$ smooth, $E(u) < \infty$. Then u is homotopic to a constant.

Corollary

M complete non–compact, ${}^M\mathrm{Ric} \geq 0$. $D \subset M$ compact, $\pi_1(\partial D) = \mathbf{1}$. Then there is no non–trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non–positive sectional curvature.

Theorem

${}^M\operatorname{Ric} \geq -k^2(x)$ *and*

$$\lambda_1\left(-{}^M\Delta - Hk^2\,(x)\right) \geq 0$$

with

- $H \geq 1$ *[Pigola-Rigoli-Setti, JFA'05]*
- $H > \frac{m-1}{m}$ *[Pigola-V., IJM]*

$\Rightarrow$ *same conclusions.*

Strategy of the proofs:

(a) an existence result for a (smooth) harmonic map with finite energy in the homotopy class of u

(b) a Liouville type theorem for finite energy harmonic maps.

f–harmonic maps

We want to reproduce the same for f–harmonic maps
$u : M \to N$ f–harmonic $\Leftrightarrow$ $u|_\Omega$ is a stationary point of the f–energy

$$E_f^\Omega(u) = \frac{1}{2}\int_\Omega |du|^2_{HS} e^{-f} dV_M$$

$\forall \Omega \subset M$ compact. When u is C^2, the Euler–Lagrange equation for the energy functional E_f is the f–harmonic maps equation

$$0 = \tau_f u := e^f \operatorname{div}(e^{-f} du) = \tau u - du(\nabla f),$$

where

- $\tau_f u$ is said f-tension field of u
- $\tau u = \operatorname{div} du$ is the standard tension field (laplacian) of u

Main result

Theorem

N compact with ${}^N\operatorname{Sect} \le 0$, M complete with ${}^M\operatorname{Ric}_f \ge -k^2(x)$. Consider $u \in C^0(M, N)$ with $E_f(u) < +\infty$. Then u is homotopic to a constant provided

$$\lambda_1(-{}^M\Delta_f - Hk^2) \ge 0 \tag{1}$$

for some $H > 1$ and at least one of the following assumptions is satisfied

(a) *there exists a constant $C > 0$ such that $|f| \le C$;*

(b) *k does not vanish identically;*

(c) *there is a point $q_0 \in M$ such that ${}^M\operatorname{Ric}_f|_{q_0} > 0$;*

Moreover if

(d) ${}^N\operatorname{Sect} < 0$,

we can conclude that u is homotopic either to a constant or to a totally geodesic map whose image is contained in a geodesic of N.

Remark.

- $\lambda_1(-{}^M\Delta_f - Hk^2) \geq 0 \Leftrightarrow$

$$H \int_M k^2 \varphi^2 e^{-f} dV_M \leq \int_M |\nabla\varphi|^2 e^{-f} dV_M, \quad \forall \varphi \in C_c^\infty(M)$$

- The case $f \equiv 0$ ($\Rightarrow$ (a)) recover [Schoen-Yau, CommMathHelv'76] and [Pigola-Rigoli-Setti, JFA'05], but $H > 1$.
- The cases (c) and (d), M compact, $m \geq 3$, are in [Lichnerowicz Symp.Math.'68].

Strategy of the proofs (as for the harmonic case):

(a) an existence result for a (smooth) f–harmonic map with finite f–energy in the homotopy class of u

(b) a Liouville type theorem for finite f–energy f–harmonic maps.

Geometric applications

Corollary

M complete with ${}^M\mathrm{Ric}_f \geq -k^2(x)$ *and* $\lambda_1(-\Delta_f - Hk^2) \geq 0$ *for some* $H > 1$.
$D \subset M$ *compact,* $\pi_1(\partial D) = 1$. *If one of the following assumptions is satisfied*

(a) *there exists a constant* $C > 0$ *such that* $|f| \leq C$,
(b) k *does not vanish identically,*
(c) *there is a point* $q_0 \in M$ *such that* ${}^M\mathrm{Ric}_f|_{q_0} > 0$,

then there is no non–trivial homomorphism of $\pi_1(D)$ *into* $\pi_1(N)$, N *compact with* ${}^N\mathrm{Sect} \leq 0$.

Moreover if

(d) ${}^N\mathrm{Sect} < 0$,

then each homomorphism of $\pi_1(D)$ *into* $\pi_1(N)$ *is either trivial or maps all* $\pi_1(D)$ *into a cyclic subgroup of* $\pi_1(N)$.

Special cases of the corollary:

Corollary (1st)

$|f|$ bounded. M complete, ${}^M\mathrm{Ric}_f \geq 0$. $D \subset M$ compact, $\pi_1(\partial D) = 1$. Then there is no non–trivial homomorphism of $\pi_1(D)$ into $\pi_1(N)$, N compact with ${}^N\mathrm{Sect} \leq 0$.

Rmk. Completely contained in the weighted Sormani's result.

On the other hand

Corollary (2nd)

M complete ${}^M\mathrm{Ric}_f \geq 0$. $D \subset M$ compact, $\pi_1(\partial D) = 1$. Then each homomorphism of $\pi_1(D)$ into $\pi_1(N)$, N compact, $\mathrm{Sect}\, N < 0$, maps all $\pi_1(D)$ into a cyclic subgroup of $\pi_1(N)$.

Rmk. No assumptions on f.

Topology of expanding solitons

Special case: M non–compact expanding gradient Ricci soliton, i.e.

$$^M\,\mathrm{Ric}_f \equiv \lambda \langle , \rangle_M , \ \ \lambda < 0.$$

It is known:

- $m\lambda \leq \inf_M \mathrm{Scal}_M \leq 0$ and $\mathrm{Scal}_M > m\lambda$, unless M is Einstein and the soliton is trivial [Pigola-Rimoldi-Setti Math.Z.'11]
- $\mathrm{Scal}_M \geq (m-1)\lambda$, $\Rightarrow$ M is connected at infinity unless M is isometric to the product $\mathbb{R} \times N$, where N compact Einstein manifold and $\mathbb{R}$ the Gaussian expanding Ricci soliton [Munteanu-Wang CAG (to app.)].

To obtain this latter

Lemma (Munteanu-Wang)

M complete non–trivial expanding gradient Ricci soliton.
Define $\rho := \mathrm{Scal}_M - m\lambda$.
Then $\rho > 0$ and $\lambda_1(-\Delta_f - \rho) \geq 0$.

If $\mathrm{Scal}_M - m\lambda > -\lambda$ then

$$\lambda_1(-\Delta_f - \rho) \geq 0 \;\Rightarrow\; \lambda_1(-\Delta_f + H\lambda) \geq 0,$$

for some $H > 1$
$\Rightarrow$ we can apply our corollary (case (a) with $k^2 = -\lambda$) to get

Corollary (3rd)

M complete non–trivial expanding gradient Ricci soliton, $\mathrm{Scal}_M > (m-1)\lambda$.
$D \subset M$ compact, $\pi_1(\partial D) = \mathbf{1}$. Then, there is no non–trivial homomorphism of $\pi_1(D)$ into $\pi_1(N)$, N compact, ${}^N\mathrm{Sect} \leq 0$.

Proof. $D \subset M$ compact, $\pi_1(\partial D) = \mathbf{1}$ and N compact with ${}^N\mathrm{Sect} \leq 0$.

- Consider a homomorphism $\beta \in \mathrm{Hom}(\pi_1(D), \pi_1(N))$.
- Since N is $K(\pi, 1)$, according to the theory of aspherical spaces there exists a map $\hat{u} : D \to N$ such that $\beta = \alpha \circ \hat{u}_\sharp$ for some automorphism $\alpha \in \mathrm{Aut}(\pi_1(N))$.
- Here $\hat{u}_\sharp : \pi_1(D) \to \pi_1(N)$ is the homomorphism induced by $\hat{u}$.
- Since $\pi_1(\partial D) = \mathbf{1}$, $\hat{u}$ can be extended to a map $u : M \to N$ such that $u|_{M \setminus D'}$ is constant for some compact set $D \subset\subset D' \subset\subset M$. Then $E_f(u) < \infty$ $\Rightarrow$ we can apply the vanishing theorem.

Case (a),(b),(c). u is homotopic to a constant $\Rightarrow$ β is trivial.

Case (d). ${}^N\mathrm{Sect} < 0$ $\Rightarrow$ u is homotopic to $v : M \to N$ s.t $Ddv = 0$ and $\mathrm{rk}(v) \equiv 1$. To conclude we need

Theorem

M, N complete Riemannian manifolds, $v \in C^2(M,N)$, $Ddv = 0$, $\mathrm{rk}(v) \equiv 1$. Then $v(M) \subseteq \gamma$, γ geodetic of N and

(1) *if γ is closed $v_\sharp(\pi_1(M,x_0)) \leq \left\langle [\gamma]_{v(x_0)} \right\rangle \leq \pi_1(N, v(x_0))$;*

(2) *if γ is not closed then $v_\sharp(\pi_1(M,x_0)) = \mathbf{1}_{\pi_1(N,v(x_0))}$.*

This theorem gives

$$\hat{u}_\sharp(\pi_1(D,x_0)) = u_\sharp(\pi_1(D,x_0)) = v_\sharp(\pi_1(M,x_0)) < \left\langle [\gamma]_{v(x_0)} \right\rangle$$

□

Proof (of the theorem). $Ddv = 0 + \mathrm{rk}(v) \equiv 1 \Rightarrow$ by standard argument there exists a geodesic γ of N such that $v(M) \subset \gamma$.
WLOG, we can take a constant speed parametrization $\gamma : \mathbb{R} \to N$.

Fix an element $\mathfrak{g} \in \pi_1(M, x_0)$. We have to compute $v_\sharp(\mathfrak{g})$.

We choose special representative $\eta : \mathbb{T}^1 = [0,1]/\sim \to M, \ [\eta] = \mathfrak{g}$ s.t. $\eta|_{(0,1)}$ is a const. sp. geodesic. In fact

- let $\sigma \in C^0([0,1], M), \ \sigma(0) = \sigma(1) = x_0, \ [\sigma] = \mathfrak{g}$
- let $\tilde{M}$ be the universal cover of M and $P_M : \tilde{M} \to M$ the covering projection
- lift σ to $\tilde{\sigma} : [0,1] \to \tilde{M}$, $\tilde{\sigma}(0), \tilde{\sigma}(1) \in P_M^{-1}(x_0)$
- choose a geodesic $\tilde{\eta} : [0,1] \to \tilde{M}$, $\tilde{\eta}(0) = \tilde{\sigma}(0)$, $\tilde{\eta}(1) = \tilde{\sigma}(1)$
- $\pi_1(\tilde{M}) = \mathbf{1} \Rightarrow \tilde{\eta} \overset{hom.}{\cong} \tilde{\sigma}$ (rel. $\{0,1\}$) $\Rightarrow \sigma \overset{hom.}{\cong} \eta := P_M \circ \tilde{\eta}$
- $\Rightarrow \eta|_{(0,1)}$ geodesic and $[\eta] = [\sigma] = \mathfrak{g}$.

- if $v \circ \eta$ is constant $\Rightarrow$ $v_\sharp([\eta]_x) = [v \circ \eta]_{v(x_0)} = \mathbf{1}_{\pi_1(N, v(x_0))}$
- if $v \circ \eta \in C^0([0,1], N)$ is not constant, then $Ddv = 0$ and $\mathrm{rk}(v) \equiv 1$ $\Rightarrow$
  - $v \circ \eta$ is a non–trivial constant speed geodesic arc in N
  - $v \circ \eta \subset \gamma$
  - $dv(\dot\eta(1)) \parallel dv(\dot\eta(0))$ and $|dv(\dot\eta(1))| = |dv(\dot\eta(0))| \neq 0$
- $\Rightarrow$ $dv(\dot\eta(1)) = dv(\dot\eta(0))$.
- $\Rightarrow$ both γ and $v \circ \eta$ are nontrivial closed geodesic of N
- $\Rightarrow$ $v \circ \eta = \gamma^s$, $s \in \mathbb{Z}$ $\Rightarrow$ $v_\sharp([\eta]_x) = [\gamma]^s_{v(x)}$.

$\square$

Existence

Theorem

$\dim M = m \geq 2$. $u : M \to N$, $E_f(u) < \infty$, N compact with ${}^N\mathrm{Sect} \leq 0$ $\Rightarrow$ in the homotopy class of u $\exists$ f-harmonic map v_0 s.t. $E_f(v_0) < E_f(u)$.

Case $f = 0$**.** Variational proof by [Burstall'84–J.London Soc]
Idea: Take $M_k \subset\subset M_{k+1}$ exhaustion of M.

- $\mathcal{H}_u := \{v : E(v) < +\infty \text{ and } (v|_{M_k})_\sharp \overset{conj}{\sim} (u|_{M_k})_\sharp \ \forall k\}$
- $v \in W^{1,2}_{loc} \Rightarrow v_\sharp : \pi_1(M) \to \pi_1(N)$ is well defined (Schoen-Yau,Burstall)
- $u \in \mathcal{H}_u \Rightarrow \mathcal{H}_u \neq \emptyset$
- N compact $\Rightarrow$ $\exists v_0 \in \mathcal{H}_u$ s.t. $E_p(v_0) = \inf\{E_p(v) : v \in \mathcal{H}_u\}$
- ${}^N\mathrm{Sect} \leq 0 \Rightarrow v_0 \in C^\infty$ (Schoen-Uhlenbeck)
- if $u \in C^0$ then ${}^N\mathrm{Sect} \leq 0 \Rightarrow N$ is $K(\pi, 1) \Rightarrow v_0$ homotopic to u

□

For $m \geq 3$, $E_f^M(u) = E(u)^{\tilde{M}}$, where $\tilde{M} = (M, e^{-\frac{2f}{m-2}} \langle \, , \, \rangle_M)$ and

$$u : M \to N \text{ is } f\text{-harmonic} \;\Leftrightarrow\; u : \tilde{M} \to N \text{ is harmonic}$$

$\Rightarrow$ easy proof.

Case $m = 2$. (f-)energy is conformal invariant $\Rightarrow$ we need a different approach ($m \geq 2$)

Let $\mathbb{T} = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\bar{M} = M \times_{e^{-f}} \mathbb{T}$, where

$$\langle \, , \, \rangle_{\bar{M}} (x, t) = \langle \, , \, \rangle_M (x) + e^{-2f(x)} dt^2.$$

Proposition

$u \in C^2(M, N)$, define $\bar{u} \in C^2(\bar{M}, N)$ as $\bar{u}(x, t) := u(x)$ for all $(x, t) \in \bar{M}$. Then

$$E^{\bar{M}}(\bar{u}) = E_f^M(u), \quad and \quad \tau\bar{u}(x, t) = \tau_f u(x).$$

In particular, u is f–harmonic $\Leftrightarrow$ $\bar{u}$ is harmonic.

f–harmonic maps on M are harmonic when trivially extended to $\bar{M} = M \times_{e^{-f}} \mathbb{T}^1$.

This is not enough: in general harmonic maps from $\bar{M}$ to N depend on t:
e.g. $M = \mathbb{R}$, $f \equiv 0$ and $N = \mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$
$\Rightarrow P : \mathbb{R} \times \mathbb{T} = \bar{M} \to N = \mathbb{T}^2$ is harmonic.

Nevertheless: Let $u : M \to N$, $E_f(u) < +\infty$
$\Rightarrow \bar{u} : \bar{M} \to N$, $E(\bar{u}) = E_f(u) < +\infty$

- Burstall $\Rightarrow \exists \bar{v} \in C^\infty(\bar{M}, N), \tau\bar{v} = 0$, which minimizes the energy in its homotopy class
- Since $\bar{v}$ is minimizer $\Rightarrow \bar{v}(x,t) = \bar{v}(x) =: v(x)$.

  In fact, let $T \in \mathbb{T}$ s.t. $E(\bar{v})|_{M\times\{T\}} \leq E(\bar{v})$. Define $\tilde{v}(x,t) = \bar{v}(x,T)$, then
  - $E(\tilde{v}) \leq E(\bar{v})$
  - $\tilde{v}$ is homotopic to $\bar{v}$
- $\Rightarrow$ v is a smooth f-minimizer $\Rightarrow$ $\tau_f v = 0$

□

Vanishing result

Theorem

N compact with ${}^N\mathrm{Sect} \leq 0$, M complete with ${}^M\mathrm{Ric}_f \geq -k^2(x)$. Consider $v \in C^\infty(M,N)$ f-harmonic with $E_f(v) < +\infty$. Then v is constant provided

$$\lambda_1(-{}^M\Delta_f - Hk^2) \geq 0 \tag{2}$$

for some $H > 1$ and at least one of the following assumptions is satisfied

(a) *there exists a constant $C > 0$ such that $|f| \leq C$;*

(b) *k does not vanish identically;*

(c) *there is a point $q_0 \in M$ such that ${}^M\mathrm{Ric}_f|_{q_0} > 0$;*

Moreover if

(d) ${}^N\mathrm{Sect} < 0$,

then v is homotopic either constant or totally geodesic with $\mathrm{rk}(v) = 1$.

Case $f = 0$. (proof of [Pigola-V. IJM])

- Bochner formula for harmonic maps gives

$$|dv|\Delta|dv| + k(x)|dv|^2 \geq |Ddv|^2 - |\nabla|dv||^2$$

- $+$ refined Kato

$$|Ddv|^2 - |\nabla|dv||^2 \geq \frac{1}{m-1}|\nabla|dv||^2.$$

- $\Rightarrow$ $\forall\eta$ cut-off compactly supported

$$\int_M \eta^2|dv|\Delta|dv| + \int_M \eta^2 k(x)|dv|^2 - \int_M \eta^2 \frac{1}{m-1}|\nabla|dv||^2 \geq 0$$

- The spectral assumption gives $\int |\nabla\varphi|^2 - H\int k(x)\,\varphi^2 \geq 0,\ \forall\varphi \in C_c^\infty$.
- $\varphi = \rho|dv| \Rightarrow$ we can deal with $k(x)$
- A suitable choice of ρ's gives a Caccioppoli inequality for v

$$\Rightarrow \int_{B(R)} |\nabla|dv||^2 \leq O(R^{-2}) \int_M |dv|^2$$

- Then

$$\begin{cases} |dv| = const. > 0 \\ \lambda_1\left({}^H L\right) \geq 0 \end{cases} \Rightarrow \begin{cases} \mathrm{Vol}(M) < \infty \\ {}^M\,\mathrm{Ric} \geq 0 \end{cases} \Rightarrow |dv| = 0$$

□

Case $f \neq 0$. Same computations, but

- We need a Bochner formula for f–harmonic maps (essentially contained in [Lichnerowicz Symp.Math.'68]):

Lemma

$v \in C^2(M,N)$. Then

$$\frac{1}{2}\Delta_f |dv|^2 = |Ddv|^2 + \langle dv, d\tau_f v\rangle_{HS} + \sum_{i=1}^m \left\langle dv({}^M\operatorname{Ric}_f(E_i,\cdot)^\sharp), dv(E_i)\right\rangle_N$$
$$- \sum_{i,j=1}^m \left\langle {}^N\operatorname{Riem}(dv(E_i), dv(E_j))dv(E_j), dv(E_i)\right\rangle_N ,$$

where $\{E_i\}_{i=1}^m$ is some chosen orthonormal frame on M.

- NO refined Kato $\Rightarrow$

$$|Ddv|^2 - |\nabla|dv||^2 \geq 0$$

- weighted version of Stokes' theorem

$$\int_M \Delta_f \phi e^{-f} dV_M = \int_M \mathrm{div}(e^{-f} d\phi) dV_M = 0, \qquad \forall \phi \in C_c^\infty(M)$$

- $\Rightarrow \ |\nabla|dv|| = |Ddv| = 0$
- $\Rightarrow$ if $|dv| = const. > 0 \ \Rightarrow \ \mathrm{Vol}_f(M) < \infty$

a) if $|f|$ is bounded, ${}^M\mathrm{Ric}_f \geq 0$ gives $\mathrm{Vol}_f(M) = \infty$;

b) if $\mathrm{Vol}_f(M) < \infty$, $\lambda_1(-\Delta_f - Hk^2) \geq 0$ gives $k \equiv 0$;

c) if $\mathrm{Ric}_f|_{q_0} > 0$, Bochner gives $|dv| \equiv 0$;

d) if ${}^N\mathrm{Sect} < 0$, Bochner gives $\mathrm{rk}(v) \equiv 1$.

□

Remarks on the case $\mathbf{f = 0}$.

- These manipulations improve also the harmonic case, by permitting $H < 1$. This weakens both SY and PRS and permits the application to minimal immersions which are δ-stable (in the sense of Colding-Minicozzi, Tam-Zhou...), i.e.
$$\lambda_1\left(-\Delta - H|\mathbf{II}|^2(x)\right) \geq 0.$$
If M minimally immersed in Q, ${}^Q\mathrm{Sect} \geq 0$, then
$${}^M\mathrm{Ric} \geq -|\mathbf{II}|^2$$
by Gauss equation.
Then, $D \subset M$ compact, $\pi_1(\partial D) = \mathbf{1} \Rightarrow$ there is no non–trivial homomorphism of $\pi_1(D)$ into $\pi_1(N)$, N compact, ${}^N\mathrm{Sect} \leq 0$.

- This technique of manipulation first used by Bérard in the field of minimal surfaces. There $|\mathbf{II}|$ (second fundamental form) instead of $|du|$ and Bochner replaced by Simons' inequality

$$|\mathbf{II}|\Delta|\mathbf{II}| + |\mathbf{II}|^4 - \frac{2}{m}|\nabla|\mathbf{II}||^2 \geq 0.$$

Theorem

Every stable oriented minimal hypersurface of $\mathbb{R}^{n+1}$ with finite total curvature is planar.

- [Bérard, PureApplMath'91] proved $n \leq 5$.
- [Shen-Zhu AmerJ'98] $\forall n$ via convergence of minimal surfaces
- Nevertheless the method of Bérard can be adapted to get an abstract proof $\forall n$ [Pigola-V.'10, preprint]

On the characterization of the f−harmonic representative:

Definition

M is said f*−parabolic* if $\exists$ $(\forall)$ $K \subset M$ compact with non–empty interior,

$$\mathrm{Cap}_f(K) = \inf\left\{E_f(\varphi)\ :\ \varphi \in C^\infty_c(M),\ \varphi|_K \geq 1\right\} = 0.$$

M is f−parabolic if every bounded above f-subharmonic function is necessarily constant, i.e.

$$\begin{cases} \Delta_f \varphi \geq 0 \text{ (weakly)} \\ \varphi \leq C < +\infty \end{cases} \quad \Rightarrow \varphi \equiv const.$$

Using $E^M_f(u) = E^{\bar{M}}\bar{u}$ we get

Lemma

Let M be a complete Riemannian manifold and $f \in C^\infty(M)$. Then M is f−parabolic if and only if $\bar{M} = M \times_{e^{-f}} \mathbb{T}$ is parabolic.

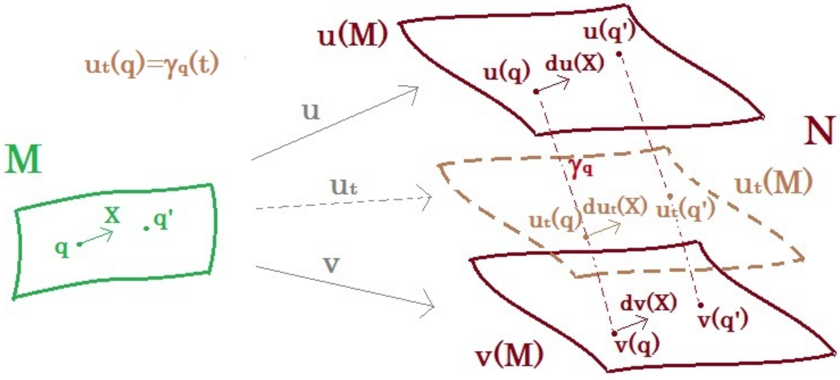
Theorem

M f–parabolic.

i) *Let $u : M \to N$ be f–harmonic, $E_f(u) < \infty$. If ${}^N\mathrm{Sect} < 0$, there's no other f–harmonic map of finite f–energy homotopic to u unless $u(M)$ is contained in a geodesic of N.*

ii) *If ${}^N\mathrm{Sect} \leq 0$ and $u, v : M \to N$ are homotopic f–harmonic maps with $E_f(u), E_f(v) < \infty$, then there is a smooth one-parameter family $u_t : M \to N$, of f–harmonic maps with $u_0 = u$ and $u_1 = v$. Moreover, for each $x \in M$, the curve $\{u_t(x) : t \in \mathbb{R}\}$ is a constant (independent of x) speed parametrization of a geodesic.*

- This is due to [Schoen-Yau, Topology'79] for $f = 0$.



Remark. If N simply connected $\Rightarrow$ $\text{dist}_N(u, v)$ is constant.

- Actually, if $\begin{cases} \pi_1(N) = \mathbf{1}, \\ M \ f\text{-parabolic} \\ v : M \to N \ f\text{-harmonic with } E_f(v) < \infty \end{cases}$ $\Rightarrow$ $v \equiv const.$