ON THE DIRICHLET PROBLEM FOR p-HARMONIC MAPS II: CARTAN-HADAMARD TARGETS WITH SPECIAL STRUCTURE

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Abstract. In this paper we develop new geometric techniques to deal with the Dirichlet problem for a p-harmonic map from a compact manifold with boundary to a Cartan-Hadamard target manifold which is either 2-dimensional or rotationally symmetric.

Introduction

Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n respectively. Let $u : M \to N$ be a C^1 map. The p-energy density $e_p(u) : M \to \mathbb{R}$ is the non-negative function defined on M as

$$e_p(u)(x) = \frac{1}{p}|du|^p_{HS}(x).$$

Here the differential du is considered as a section of the $(1, 1)$-tensor bundle along the map u, i.e. $du \in \Gamma(T^*M \otimes u^{-1}TN)$ is a vector valued differential 1-form. Moreover $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product. If $\Omega \subset M$ is a compact domain, we define the p-energy of $u|_\Omega : \Omega \to N$ by

$$E^\Omega_p(u) = \int_\Omega e_p(u) dV_M.$$

Let X be a C^1 vector field along u, i.e. a section of the bundle $u^{-1}TN$, supported in Ω. Then

$$u_t(x) = {}^N\exp_{u(x)} tX(x).$$

defines a variation of u which preserves u on $\partial\Omega$. The map $u : M \to N$ is said to be p-harmonic if, for each compact domain $\Omega \subset M$, it is a stationary point of the p-energy functional, that is

$$\left.\frac{d}{dt}\right|_{t=0} E^\Omega_p(u_t) = \int_M \left\langle |du|^{p-2}du, dX\right\rangle_{HS} dV_M = 0.$$

The latter equality corresponds to the weak formulation of the p-Laplace equation

$$\Delta_p u = \operatorname{div}(|du|^{p-2}du) = 0. \tag{1}$$

Here $-\operatorname{div} = \delta$ is the formal adjoint of the exterior differential d, with respect to the standard L^2 inner product on vector-valued differential 1-forms on M.

The theory of p-harmonic maps between Riemannian manifolds and p-energy minimizers has undergone a great development in the last two decades. Among the works on the subject, let us recall for instance [DF, DGK, Fu1, Fu2, Ga, HL, N, XY], dedicated to the regularity theory, and [Wh, DF2, PRS, T, We2, Ma, PV1] which are concerned mostly with the connections to the geometry of the manifolds.

In [PV2], extending previous results in the literature we gave a complete solution to the homotopic p-Dirichlet in case the target manifold N is compact. The proof therein is purely variational. Exploiting powerful techniques due to B. White, [Wh], one can define the weak relative d-homotopy type of $W^{1,p}$ maps, hence minimize the p-energy in the d-homotopy class of the initial datum, and finally show how to apply R. Hardt and F.-H. Lin's regularity theory to the minimizer, [HL].

In this paper we focus our attention on a non-compact, but topologically trivial, target manifold N of non-positive curvature. In this setting, important contributions have been given by Fuchs, [Fu1, Fu2], who adapted to the $p > 2$ case the analytic strategy introduced by Hildebrandt, Kaul and Widman for harmonic maps, [HKW3]. See also the more recent [FR3]. One crucial point in these works is a quite implicit use of a tight relation between two different notions of bounded Sobolev maps: a first one, that we could call *intrinsic*, is defined in a global coordinate chart of the target space. A second one, somewhat more standard and called *extrinsic*, uses a proper isometric embedding of the target into a Euclidean space of sufficiently large dimension. In a future paper, [PV3], we shall investigate carefully the relations between these two notions and we will point out some interesting consequences. Our main purpose, here, is to show how some new geometric constructions, concerning manifolds of non-positive curvature, can help the solution of the purely analytic original problem. Actually, we feel that our approach, which is based on a compactification procedure, will be useful in more general settings where the analytic problem is related to different functionals. Indeed, as it will be clear from the proof, the relevant properties of the p-energy required by the method we propose are: (a) the solvability of the problem when the target is compact and (b) a maximum principle for regular enough solutions. On the other hand, some of the techniques presented here will be used in a forthcoming paper, [PV3], to face the p-Dirichlet problem for maps with values in a convex geodesic ball of a generic target space.

The starting point of the present investigation is that the only interesting case involves target manifolds without compact quotients for, otherwise, the non-compact problem can be reduced to the compact one where the machinery alluded to above can be applied without changes.

Theorem A. *Let (M,g) be a compact, m-dimensional Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and let (N,h) be a complete, Riemannian manifold of dimension n such that its universal cover supports a strictly convex exhaustion function. Assume that there exists a subgroup Γ of isometries of N acting freely, properly and co-compactly on N. Then, for any $p \geq 2$ and for every $f \in C^0(M,N) \cap Lip(\partial M, N)$, the homotopy p-Dirichlet problem has a solution $u \in C^{1,\alpha}(\mathrm{int}(M),N) \cap C^0(M,N)$. Moreover, the solution is unique provided N has non-positive sectional curvature.*

We aim at facing the general situation where either we have no information on the structure of the isometry group of N or it is known that N has no compact quotients. This latter case occurs, for instance, if its geometry is not bounded at some finite order. As a first step forward in this direction, we decide to focus our attention on rotationally symmetric Cartan-Hadamard targets, i.e., we add the request that the complete non-positively curved manifold N is simply connected, hence it is diffeomorphic to the Euclidean space via the exponential map (from any reference point), and that its metric tensor, in a global polar coordinates system, is rotational symmetric around the origin of the system. Obviously, the homotopy condition is trivially satisfied by any continuous solution of the p-Dirichlet problem.

Formally, having fixed a smooth function $\sigma : [0,+\infty) \to [0,+\infty)$ satisfying

$$\sigma^{(2k)}(0) = 0, \forall k \in \mathbb{N}, \quad \sigma'(0) = 1, \quad \sigma(r) > 0, \forall r > 0, \tag{2}$$

we shall denote by N^n_σ the smooth n-dimensional Riemannian manifold given by

$$\left([0,+\infty) \times \mathbb{S}^{n-1}, dr^2 + \sigma^2(r)\, d\theta^2\right), \tag{3}$$

where $d\theta^2$ denotes the standard metric on $\mathbb{S}^{n-1}$. Clearly, N^n_σ is diffeomorphic to $\mathbb{R}^n$ and geodesically complete for any choice of σ. Usually, N^n_σ is called a model manifold with warping function σ and pole 0. The r-coordinate in the expression (3) of the metric represents the distance from the pole. Thus, at a given point of N^n_σ we distinguish the radial sectional curvatures and the tangential sectional curvatures of the model, according to whether the 2-plane at hand contains the radial vector field $\partial/\partial r$ or not. Standard formulas for warped product metrics reveal that

$$\mathrm{Sect}_{rad} = -\frac{\sigma''}{\sigma}, \quad \mathrm{Sect}_{tg} = \frac{1-(\sigma')^2}{\sigma^2} \tag{4}$$

Thus, in particular, the model manifold N^n_σ is Cartan-Hadamard if and only if

$$\sigma'' \geq 0.$$

Indeed, since $\sigma'(0) = 1$, the convexity of σ always implies $\sigma' \geq 1$. It is worth to point out that there are Cartan-Hadamard model manifolds with bounded (pinched) negative curvature and without compact quotients. An example of special geometric interest was constructed by M. Anderson, [An], to settle in the negative a conjecture due to J. Dodziuk on the L^2-cohomology in pinched negative curvature.

The second main result of the paper is represented by the following

Theorem B. *Let* (M,g) *be a compact,* m*-dimensional Riemannian manifold with smooth boundary* $\partial M \neq \emptyset$ *and let* N *be an* n*-dimensional model manifold of non-positive curvature* N^n_σ*,* $n > 3$*. Then, for any* $p \geq 2$ *and any given* $f \in C^0(M,N) \cap Lip(\partial M, N)$*, the* p*-Dirichlet problem*

$$\begin{cases} \Delta_p u = 0 & \text{on } M \\ u = f & \text{on } \partial M, \end{cases} \tag{5}$$

has a unique solution $u \in C^{1,\alpha}(\mathrm{int}(M), N) \cap C^0(M)$.

We point out that, when the Cartan-Hadamard target is 2-dimensional, the first equation in (4) defines its Gaussian curvature in polar coordinates regardless of any rotational symmetry condition. Namely, given a 2-dimensional Cartan-Hadamard manifold (N, h_N), in the global geodesic chart (r,θ) around some fixed pole $o \in N$ the metric h_N can be expressed as

$$h_N|_{(r,\theta)} = dr^2 + \nu^2(r,\theta)d\theta^2.$$

Since the function $\nu > 0$ completely determines the Riemannian structure of N, in the following we will use the notation $N = N^2_\nu$. Direct computations show that the only (radial) sectional curvature of N^2_ν satisfies at any point (r,θ) the formula

$$\mathrm{Sect}(r,\theta) = \mathrm{Sect}_{rad}(r,\theta) = -\nu^{-1}(r,\theta)\frac{\partial^2\nu(r,\theta)}{\partial r^2}. \tag{6}$$

Accordingly, the same ideas we will use to prove Theorem B, permit to obtain also the following

Theorem C. *Let* (M,g) *be a compact,* m*-dimensional Riemannian manifold with smooth boundary* $\partial M \neq \emptyset$ *and let* N *be a Cartan-Hadamard 2-dimensional manifold* N^2_ν*. Then, for any* $p \geq 2$ *and any given* $f \in C^0(M,N) \cap Lip(\partial M, N)$*, the* p*-Dirichlet problem* (5) *has a unique solution* $u \in C^{1,\alpha}(\mathrm{int}(M), N) \cap C^0(M)$*.*

The approach we propose inspires to the reduction procedure used to obtain Theorem A. This latter implies that the Dirichlet problem is easily solved when N has a compact quotient, but, as observed above, this is not the case for a general Cartan-Hadamard model manifold N^n_σ. The possible lack of discrete, co-compact isometry subgroups is overcome by using a combination of cut&paste and periodization arguments. Namely, we will show that it is possible to perturb the metric of N^n_σ in the exterior of a fixed geodesic ball in N^n_σ such that the complete manifold thus obtained is again Cartan-Hadamard and has compact quotients. A new maximum principle for the composition of the p-harmonic map and the convex distance function of N^n_σ then gives that this perturbation does not affects the solution to the original problem. The uniqueness part of the theorem can be clearly considered as a bypass product of the reduction to the compact case. We recall also that a comprehensive uniqueness result for general complete targets with non-positive curvature was obtained in [PV2].

Actually, motivated by the known results [HKW1, HKW2, HKW3, Fu2], one may wonder whether the strategy outlined above can be adapted when the Cartan-Hadamard target is replaced by a convex ball. In this respect, it is worth to point out that the perturbation procedure outside a large ball can be carried out in such a way that the resulting manifold supports a strictly convex exhaustion function, see Section 3, and this kind of construction, with a suitable choice of the perdurbed spaces, looks very promising to produce a solution of the p-Dirichlet problem in the more general setting alluded to above, [PV3]

The paper is organized as follows. We will begin by proving Theorem A, which permits to solve directly the relative homotopy Dirichlet problem for p-harmonic maps when the target manifold admits a compact quotient. The remaining part of the paper aims to prove Theorem B by reproducing a similar argument even in case the target Cartan-Hadamard space N does not possess compact quotients. This will be done in Section 6. The proof relies on the preliminary results collected in Sections 2–5. In particular, in Sections 2 and 3 we discuss a procedure to glue a large ball of N with a hyperbolic space of sufficiently negative curvature $-k \ll -1$. In Section 4 we record that hyperbolic spaces have discrete, co-compact groups of isometries with arbitrarily large fundamental domains. Finally, in Section 5 we introduce a new maximum principle for the composition of a p-harmonic map and a convex function. The minor changes in the proof of Theorem B needed to obtain Theorem C are detailed in Section 2.

1. Manifolds with compact quotients

Even if the proof is pretty elementary, the ideas contained there will be the basis for the periodization procedure developed in the following subsections to prove Theorems B and C.

Proof (of Theorem A). By assumption, $N' = N/\Gamma$ is a compact, aspherical Riemannian manifold covered by N via the quotient projection $P : N \to N'$. The original datum f projects to a new function $P(f) : M \to N'$ which, in turn, can be used to state the corresponding p-Dirichlet problem

$$\begin{cases} \Delta_p u' = 0 & \text{on } M \\ u' = P(f) & \text{on } \partial M. \end{cases}$$

Thanks to the analysis of the compact target case provided in [PV2], this problem admits a solution $u' \in C^{1,\alpha}(\mathrm{int}(M), N') \cap C^0(M, N')$ in the homotopy class of $P(f)$ relative to ∂M. Let $H' : [0,1] \times M \to N'$ be such a homotopy. The classical theory of fibrations (see e.g. [Hat]) then tells us that H' lifts to a homotopy $H : [0,1] \times M \to N$ satisfying $H(1,x) = f(x)$. The homotopy H is relative to ∂M because, for every $y \in \partial M$, $H([0,1] \times \{y\})$ is contained in the (discrete) fibre over $P(f)(y)$. Let $u(x) = H(0,x)$. Since P is a local isometry and $P(u) = u'$, then u is p-harmonic in M of class $C^{1,\alpha}(\mathrm{int}(M), N) \cap C^0(M,N)$. On the other hand, using the fact that H is relative to ∂M we deduce that $u = f$ on ∂M. This proves that the original homotopy p-Dirichlet problem has a solution. In case ${}^N\mathrm{Sect} \leq 0$, uniqueness follows easily from the following few facts: (a) solutions of the homotopy p-Dirichlet problem with target N projects to solutions of the corresponding problem with target N'; (b) in case of compact targets, the solution is unique; (c) liftings are uniquely determined by their values at a single point. □

2. Gluing model manifolds keeping $\mathrm{Sect} \leq 0$

In this Section we show that, in some sense, it is possible to prescribe a hyperbolic infinity to a Cartan-Hadamard model, as well as to a generic Cartan-Hadamard 2-manifold, without violating the non-positive curvature condition. It is convenient to put the following

Definition 2.1. *By a model of hyperbolic type we mean a model manifold H^m_σ whose warping function satisfies the following further requirements:*

(i) $\sigma'' \geq 0$.
(ii) *Let* $\sigma_k(r) = k^{-1/2}\sigma\left(k^{1/2}r\right)$. *Then, for every* $r > 0$,

$$\sigma_k'(r) \geq \sigma_k(r) \to +\infty, \text{ as } k \to +\infty.$$

Note that a hyperbolic type model is Cartan-Hadamard and has, at least, an exponential volume growth. Clearly, the choice $\sigma(r) = \sinh(r)$ is admissible and the corresponding model is the standard hyperbolic spaceform. Whence, the choice of the name. Note also that $H^m_{\sigma_k} = k^{-1} H^m_\sigma$ in the Riemannian sense.

We are going to show that every compact ball centered at the pole of a model of non-positive curvature can be glued to a hyperbolic type model thus giving a new model manifold with non-positive curvature.

Theorem 2.2. *Let N^n_ρ be a Cartan-Hadamard model and let H^n_σ be of hyperbolic type. Fix $\bar{R} > 0$. Then, for every $R > \bar{R}$ there exist a $k = k(R) >> 1$ and a Cartan-Hadamard model M^n_τ such that:*

(i) $B^N_{\bar{R}}(0) \subset M^n_\tau$.
(ii) $M^n_\tau \setminus B^M_R(0) = H^n_{\sigma_k} \setminus B^H_R(0)$.

Proof. Thanks to (4), it is enough to produce a warping function $\tau : [0,+\infty) \to [0,+\infty)$ satisfying the following requirements:

(a) $\tau = \rho$ on $[0,R)$.
(b) $\tau = \sigma_k$ on $(R,+\infty)$, $R >> 1$.

(c) $\tau' \geq 1$ and $\tau'' \geq 0$ on $[0, +\infty)$.

To this end, let $\bar{R} < R_1 < R_2$. By the assumptions on σ, we can choose $k = k(R_1, R_2) > 0$ large enough so that

$$\rho'(R_1) \leq \frac{\sigma_k(R_2) - \rho(R_1)}{R_2 - R_1} \leq \sigma_k'(R_2). \tag{7}$$

Define

$$\tau_1(r) = \begin{cases} \rho(r) & \text{on } [0, R_1) \\ \rho(R_1) + \frac{\sigma_k(R_2) - \rho(R_1)}{R_2 - R_1} r & \text{on } [R_1, R_2] \\ \sigma_k(r) & \text{on } (R_2, +\infty). \end{cases}$$

Then, τ_1 is a piecewise smooth, convex function with $\tau_1' \geq 1$. To complete the construction of τ, it remains to smoothing out the angles with a convex function. This can be done using the approximation procedure described by M. Ghomi in [Gh]. □

Thanks to the explicit formula (6), in a completely analogous way we can obtain also the following

Theorem 2.3. *Let N_ν^2 be a 2-dimensional Cartan-Hadamard manifold and let H_σ^2 be a 2-dimensional manifold of hyperbolic type. Fix $\bar{R} > 0$. Then, for every $R > \bar{R}$ there exist a $k = k(R) >> 1$ and a Cartan-Hadamard manifold M_τ^2 such that:*

(i) $B_{\bar{R}}^N(0) \subset M_\tau^n$.

(ii) $M_\tau^2 \setminus B_R^M(0) = H_{\sigma_k}^2 \setminus B_R^H(0)$.

Remark 2.4. *As it is clear from the proof, Theorem 2.2 and Theorem 2.3 hold for a class of "external" manifolds wider than the class of models of hyperbolic type. Namely, the condition (ii) in Definition 2.1 is stronger than necessary, since what is only needed is relation (7) to hold.*
For instance, one can chose $\sigma(r)|_{(R_2,+\infty)} = \alpha r - C$, for large enough constants $\alpha = \alpha(\rho) > 1$ and $C = C(\rho) > 0$. This non-trivial example has linear volume growth and its sectional curvatures satisfy $\mathrm{Sect}_{rad} = 0$ and $\mathrm{Sect}_{tg} = -\frac{\alpha-1}{\alpha^2} r^{-2}$ for $r > R_2$.

3. Convex exhaustion functions on glued manifolds

As a consequence of the Hessian comparison theorem, the square of the distance function of a generic Cartan-Hadamard manifold M is a smooth, strictly convex exhaustion function. This Section aims to show how it is possible to prescribe an hyperbolic infinity to M in such a way that the resulting space supports again a strictly convex, exhaustion function. The lack of rotational symmetry of the source metric of M will prevent us to guarantee that the new space is Cartan-Hadamard. On the other hand, we are not able to guarantee that the space supporting the strictly convex function has a compact quotient. Therefore, this construction will be not used in the proof of the main theorem of the paper. However, we feel that it is interesting in its own and represents a first important indication that the p-Dirichlet problem can be solved in the more general situation where the target space is simply Cartan-Hadamard.

Let $(N_j, \langle , \rangle_{N_j})$ be a n-dimensional Cartan-Hadamard manifold and fix a pole $o \in N_j$. Consider $\mathbb{R}^n$ with polar coordinates (t, Θ) around o. Namely, in a neighborhood of each point $x \in \mathbb{R}^n$ we have a local coordinate system $(t, \theta^2, \dots, \theta^n)$, where t is the radial coordinate and $(\theta^i)_{i=2}^n$ are local angular coordinates. Since N_j is Cartan-Hadamard, by the Gauss Lemma we can write

$$(N_j, \langle , \rangle_{N_j}) = (\mathbb{R}^n, dt^2 + j_{il}(t, \Theta) d\theta^i d\theta^l).$$

Similarly, we consider the hyperbolic space $\mathbb{H}^n_k$ of constant curvature $-k$, and we write $\mathbb{H}^n_k = (\mathbb{R}^n, dt^2 + h^{(k)}_{il} d\theta^i d\theta^l)$.

Theorem 3.1. *Let N^n_j be an n-dimensional Cartan-Hadamard manifold. Fix $0 < R_1 < R_2 < \infty$. Then, there exist $k > 0$ depending on j, R_1 and R_2, and a manifold $(N^n_{\hat{j}}, \langle , \rangle_{N_{\hat{j}}}) = (\mathbb{R}^n, dt^2 + \hat{j}_{il}(t, \Theta) d\theta^i d\theta^l)$ such that:*

(i) $B^{N_j}_{R_1}(o) \subset N^n_{\hat{j}}$.
(ii) $N^n_{\hat{j}} \setminus B^{N_{\hat{j}}}_{R_2}(\hat{o}) = \mathbb{H}^n_k \setminus B^{\mathbb{H}^n_k}_{R_2}(o')$ *for some poles $\hat{o} \in N_{\hat{j}}$ and $o' \in \mathbb{H}^n_k$.*
(iii) $N_{\hat{j}}$ *supports a global strictly convex function.*

Proof. Consider a smooth partition of unity $\phi_j, \phi_h \in C^\infty((0, +\infty))$ such that

$$0 \le \phi_j(t) \le 1, \quad \phi_j|_{(0,R_1]} \equiv 1, \quad \phi_j|_{[R_2,\infty)} \equiv 0, \quad \phi_j' \le 0$$

and

$$\phi_j(t) + \phi_h(t) = 1, \quad \forall t \in (0, +\infty). \tag{8}$$

Define a new Riemannian manifold $N^n_{\hat{j}}$, by endowing $\mathbb{R}^n$ with a new metric $\langle , \rangle_{N_{\hat{j}}}$. Namely, in polar coordinates, we set

$$N_{\hat{j}} = (\mathbb{R}^n, dt^2 + \hat{j}_{il}(t, \Theta) d\theta^i d\theta^l),$$

where

$$\hat{j}_{il}(t, \Theta) := \phi_j(t) j_{il}(t, \Theta) + \phi_h(t) h^{(k)}_{il}(t, \Theta).$$

Note that $N_{\hat{j}}$ is a well defined n-dimensional Riemannian manifold and conditions (i) and (ii) of the statement are automatically satisfied by construction. Define the coordinate function $T : \mathbb{R}^n \to \mathbb{R}$ as $T(t, \Theta) = t$ and observe that $T \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Moreover, since N_j and $\mathbb{H}^n_k$ are Cartan-Hadamard manifolds, we have that T on N_j and $\mathbb{H}^n_k$ is the Riemannian radial function. In particular T is convex both on N_j and $\mathbb{H}^n_k$, strictly convex off the radial direction, and T^2 is smooth and strictly convex on both N_j and $\mathbb{H}^n_k$. To prove the theorem we will show that $T : N_{\hat{j}} \to \mathbb{R}$ is convex (strictly except for the radial direction). This will imply that $T^2 : N_{\hat{j}} \to \mathbb{R}$ is strictly convex and, because of (i), smooth on all of $N_{\hat{j}}$. To this end, we use the following lemma.

Lemma 3.2. *Consider a Riemannian manifold structure N_j on $\mathbb{R}^n \setminus \{0\}$, i.e. $N_j = (\mathbb{R}^n \setminus \{0\}, \langle , \rangle_{N_j})$, and suppose that, in (local) polar coordinates and with notation as above, $\langle , \rangle_{N_j}$ can be expressed as*

$$\langle , \rangle_{N_j} |_{(t,\Theta)} = dt^2 + j_{il}(t,\Theta) d\theta^i d\theta^l. \tag{9}$$

Define $T : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ as $T(t,\Theta) = t$. Then

$$^{N_j}\operatorname{Hess} T|_{(t,\Theta)}(X,X) = \frac{1}{2} X^i X^l \frac{\partial}{\partial t} j_{il}(t,\Theta),$$

for all vector fields on $\mathbb{R}^n$.

Proof (of Lemma 3.2). By definition of hessian, it holds

$$\begin{aligned} ^{N_j}\operatorname{Hess} T|_{(t,\Theta)}(X,X) &= \left\langle ^{N_j}\nabla_X {}^{N_j}\nabla T, X \right\rangle_{N_j} \\ &= \frac{1}{2} {}^{N_j}\nabla T \left\langle X, X \right\rangle_{N_j} + \left\langle X, [X, {}^{N_j}\nabla T] \right\rangle_{N_j}, \end{aligned}$$

for any vector field X on $\mathbb{R}^n$. Since T is defined as the coordinate function t and the metric has the expression (9), we have $^{N_j}\nabla T = \frac{\partial}{\partial t} =: \partial_t$. Given a vector field X, this can be expressed in coordinates as $X = X^0 \partial_t + X^i \partial_i$, where $\partial_i := \frac{\partial}{\partial \theta^i}$ for $i = 2, \dots, n$. Whence, we get

$$\begin{aligned} \frac{1}{2} {}^{N_j}\nabla T \left\langle X, X \right\rangle_{N_j} &= \frac{1}{2} \partial_t \left(\left(X^0\right)^2 + j_{il}(t,\Theta) X^i X^l \right) \\ &= X^0 \partial_t X^0 + j_{il}(t,\Theta) X^i \partial_t X^l + \frac{1}{2} X^i X^l \frac{\partial}{\partial t} j_{il}(t,\Theta). \end{aligned} \tag{10}$$

Moreover

$$\left[X, {}^{N_j}\nabla T\right] = [X, \partial_t] = -\partial_t X^0 \partial_t - \partial_t X^i \partial_i,$$

which gives

$$\begin{aligned} \left\langle X, [X, {}^{N_j}\nabla T] \right\rangle_{N_j} &= \left\langle X^0 \partial_t + X^i \partial_i, -\partial_t X^0 \partial_t - \partial_t X^i \partial_i \right\rangle_{N_j} \\ &= -X^0 \partial_t X^0 - j_{il}(t,\Theta) X^i \partial_t X^l. \end{aligned} \tag{11}$$

Summing (10) and (11) concludes the proof. □

According to Lemma 3.2 we thus have

$$\begin{aligned} ^{N_{\hat{j}}}\operatorname{Hess} T|_{(t,\Theta)}(X,X) &= \frac{1}{2} X^i X^l \partial_t \hat{j}_{il}(t,\Theta) \\ &= \frac{1}{2} X^i X^l \partial_t \left[\phi_j(t) j_{il}(t,\Theta) + \phi_h(t) h^{(k)}_{il}(t,\Theta) \right] \\ &= \phi_j(t) \left[\frac{1}{2} X^i X^l \partial_t j_{il}(t,\Theta) \right] \\ &+ \phi_h(t) \left[\frac{1}{2} X^i X^l \partial_t h^{(k)}_{il}(t,\Theta) \right] \\ &+ \partial_t \phi_j \frac{1}{2} X^i X^l j_{il}(t,\Theta) + \partial_t \phi_h \frac{1}{2} X^i X^l h^{(k)}_{il}(t,\Theta). \end{aligned}$$

Applying again Lemma 3.2 and recalling (8), this latter gives

$$\begin{aligned} {}^{N_{\hat{j}}}\operatorname{Hess} T|_{(t,\Theta)}(X,X) = \phi_j(t)^{N_j} \operatorname{Hess} T|_{(t,\Theta)}(X,X) \\ + \phi_h(t)^{\mathbb{H}^n_k} \operatorname{Hess} T|_{(t,\Theta)}(X,X) \\ + \frac{1}{2}\phi_h'(t) X^i X^l \left[h^{(k)}_{il}(t,\Theta) - j_{il}(t,\Theta) \right]. \end{aligned}$$

Since $\phi_h' \geq 0$ and recalling the above considerations, in order to conclude the proof it's enough to show that for k large enough it holds

$$X^i X^l h^{(k)}_{il}(t,\Theta) \geq X^i X^l j_{il}(t,\Theta)$$

for all $(t,\Theta) \in \mathbb{B}_{R_2} \setminus \mathbb{B}_{R_1} \subset \mathbb{R}^n$. Since $\bar{\mathbb{B}}_{R_2} \setminus \mathbb{B}_{R_1}$ is compact, there exists a constant $c_2 = c_2(R_1, R_2) > 0$ such that

$$X^i X^l h^{(1)}_{il}(t,\Theta) \geq c_2 X^i X^l j_{il}(t,\Theta)$$

for all vector fields X and all $(t,\Theta) \in \mathbb{B}_{R_2} \setminus \mathbb{B}_{R_1}$. Finally, since the coordinate system is fixed, we have that

$$h^{(k)}_{il} = \frac{\sinh^2\left(\sqrt{k}t\right)}{k \sinh^2 t} h^{(1)}_{il},$$

so that it is enough to choose k in such a way that

$$\sinh^2\left(\sqrt{k}R_1\right) \geq c_2^{-1} k \sinh^2 R_1.$$

□

Remark 3.3. In the proof of Theorem 3.1 the assumption ${}^{N_j}\operatorname{Sect} \leq 0$ was required in order to guarantee that the metric of N_j has the form (9) and that T is strictly convex in $B_{R_2} \subset N_j$. Accordingly, it is clear that the Theorem works as well when N_j is the interior of a convex geodesic ball without curvature assumptions.

Theorem 3.4. *Let $\mathcal{B}_R \subset N^n$ be a convex geodesic ball of radius R in an n-dimensional manifold. Then, for every $0 < R_1 < R_2 < R$ there exist a $k > 0$ depending on N, R_1 and R_2, and a manifold $(N^n_{\hat{j}}, \langle,\rangle_{N_{\hat{j}}}) = (\mathbb{R}^n, dt^2 + \hat{j}_{il}(t,\Theta) d\theta^i d\theta^l)$ such that:*

(i) $B^N_{R_1}(o) \subset N^n_{\hat{j}}$.

(ii) $N^n_{\hat{j}} \setminus B^{N_{\hat{j}}}_{R_2}(\hat{o}) = \mathbb{H}^n_k \setminus B^{\mathbb{H}^n_k}_{R_2'}(o')$ *for some poles $\hat{o} \in N_{\hat{j}}$ and $o' \in \mathbb{H}^n_k$.*

(iii) $N_{\hat{j}}$ *supports a global strictly convex exhaustion function.*

4. Compact hyperbolic manifolds with large injectivity radii

It is intuitively clear that actions of small discrete groups on a complete Riemannian manifold give rise to large fundamental domains. The intuition is confirmed in the next simple result.

Lemma 4.1. *Let (N,h) be a complete Riemannian manifold. Suppose that there exists a filtration*

$$\Gamma_0 \rhd \Gamma_2 \rhd \Gamma_3 \rhd \cdots \rhd \Gamma_k \rhd \cdots \rhd \{1\}$$

of discrete groups $\Gamma_k \subset \mathrm{Iso}\,(N)$ acting freely and properly on N. Then, for every arbitrarily large ball $B_R\,(p)$, there exists $K > 0$ such that the following holds: for every $k > K$ we find a fundamental domain Ω_k of Γ_k containing p and satisfying

$$B_R^N\,(p) \subset\subset \Omega_k. \tag{12}$$

Proof. Let $D_k\,(p)$ be the Dirichlet domain of Γ_k centered at p. Recall that $D_k\,(p) = \cap_{\gamma\in\Gamma_k} H_\gamma\,(p)$ where

$$H_\gamma\,(p) = \{x \in N : d_N\,(x,p) < d_N\,(x, \gamma\cdot p)\}\,.$$

One can easily verify that if $B_R^N\,(p) \cap (N \backslash D_k\,(p)) \neq \emptyset$ then

$$B_R^N\,(p) \cap \gamma \cdot B_R^N\,(p) \neq \emptyset, \tag{13}$$

for some $\gamma \in \Gamma_k \subset \Gamma_0$. Since Γ_0 acts properly on N it follows that (13) can be satisfied for at most a finite number of $\gamma_1, ..., \gamma_N \in \Gamma_0$. To conclude the validity of (12), we now use that $\cap\Gamma_k = \{1\}$ and, therefore, $\gamma_1, ..., \gamma_N \notin \Gamma_k$, for every large enough k. □

There are situations where condition (12) has an immediate interpretation in terms of the injectivity radius of the corresponding quotient space. Let N be a Cartan-Hadamard manifold and let $\Gamma \subset \mathrm{Iso}\,(N)$ be a discrete group acting freely and co-compactly on N. Then, the orbit space $N_\Gamma = N/\Gamma$ is a smooth manifold universally covered by the quotient projection $P : N \to N_\Gamma$ and the metric of N descends to a complete metric on N_Γ. Moreover $\Gamma \simeq \pi_1\,(N_\Gamma)$. By the Cartan-Hadamard theorem, another way to define the universal covering of N_Γ is to use the exponential map $\exp_q : T_q N_\Gamma \to N_\Gamma$ from a fixed point $q \in N_\Gamma$. The universal covering property yields that there exists a fiber-preserving diffeomorphism $I : N \to T_q N_\Gamma$. Therefore, we can always identify $N = T_q N_\Gamma$ and $P = \exp_q$. Fix $p \in N$ and let $q = P\,(p)$. Let also $\mathcal{E}_q\,(N_\Gamma) = N_\Gamma \backslash \,\mathrm{cut}\,(q)$ and $\Omega = \exp_q^{-1} \mathcal{E}_q$. Then Ω is a fundamental domain for the action of Γ on N and $p \in \Omega$. In particular, from the equality $P(B_R^N\,(p)) = B_R^{N_\Gamma}\,(q)$ we deduce that, for every $R < \mathrm{inj}_{N/\Gamma}\,(q)$, it holds $B_R^N\,(p) \subset\subset \Omega$. Summarizing, on a Cartan-Hadamard manifold, the existence of a co-compact discrete group of isometries with large fundamental domain follows from the existence of a quotient manifold with large injectivity radius. The converse also holds because $\mathrm{inj}\,(N) = +\infty$. Therefore, if $B_R^N\,(p) \subset\subset \Omega$ where Ω is a fundamental domain of the action, since P is an isometry on Ω it follows that $B_R^{N_\Gamma}\,(q) = P(B_R^N\,(p))$ is inside the cut-locus of q, i.e., $\mathrm{inj}\,(q) \geq R$.

A case of special interest is obtained by taking $N = \mathbb{H}^n_{-k^2}$, the standard hyperbolic spaceform of constant curvature $-k^2 < 0$. If Γ is a co-compact discrete group of isometries acting freely and properly on $\mathbb{H}^n_{-k^2}$, the corresponding Riemannian orbit space $\mathbb{H}^n_{-k^2}/\Gamma$ is named a compact hyperbolic manifold (of constant curvature $-k^2$). The following result was first observed in [Fa], see p.74.

Proposition 4.2. *Let $n \geq 0$, $R > 0$ and $p \in \mathbb{H}^n_{-k^2}$. Then, there exists a co-compact, discrete group Γ of isometries of $\mathbb{H}^n_{-k^2}$ acting freely and properly on $\mathbb{H}^n_{-k^2}$ and whose fundamental domain Ω containing p satisfies*

$$\mathbb{B}_R(p) \subset\subset \Omega.$$

Equivalently,

$$\operatorname{inj}\left(\mathbb{H}^n_{-k^2}/\Gamma\right) \geq R.$$

Proof. By a result of A. Borel [Bo], $\mathbb{H}^n_{-k^2}$ has a co-compact, discrete group of isometries Γ_0 acting freely and properly. According to a result by A. Malcev, Γ_0 is residually finite, i.e., there exists a filtration

$$\Gamma_0 \rhd \Gamma_2 \rhd \Gamma_3 \rhd \cdots \rhd \Gamma_k \rhd \cdots \rhd \{1\}$$

satisfying $[\Gamma_k : \Gamma_{k-1}] = |\Gamma_k/\Gamma_{k-1}| < +\infty$. To conclude, we now apply Lemma 4.1 and recall the previous discussion on the injectivity radius. ☐

5. A maximum principle for p-harmonic maps

It is well known, and an easy consequence of the composition law of the Hessians, that by composing a harmonic map $u : M \to N$ with a convex function $h : N \to \mathbb{R}$ gives a subharmonic function $v = h \circ u : M \to \mathbb{R}$, i.e., $\Delta v \geq 0$. In particular, if M is compact with smooth boundary $\partial M \neq 0$ and N is Cartan-Hadamard, we can choose $h(x) = d^2_N(x, o)$ and apply the usual maximum principle to conclude that the image $u(M) \subset N$ is confined in a ball $B^N_R(o)$ of radius $R > 0$ depending only on the values of u on $\partial\Omega$, namely, $R = \max_{\partial\Omega} d_N(u, o)$. Very recently, it was proved in [Ve1] that, in general, the nice composition property of harmonic maps does not extend to p-harmonic maps, $p > 2$. Nevertheless, we are able to recover the above conclusion thus establishing a new maximum principle for the composition of a p-harmonic map and a convex function.

Theorem 5.1. *Let M be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$, and let $u \in C^1(M, N)$ be a $p\,(\geq 2)$-harmonic map. Assume that N supports a smooth convex function $f : N \to \mathbb{R}$. Set $w = f \circ u : M \to \mathbb{R}$. Then*

$$\sup_M w = \sup_{\partial M} w.$$

Remark 5.2. In view of the maximum principle 5.1, one would be tempted (and it would be more natural) to minimize the p-energy directly on a bounded region of the Cartan-Hadamard target, in order to prove Theorem B avoiding the periodization procedure. Nevertheless the proof of Theorem 5.1 seems to require the p-harmonic map to be a priori at least C^1 and we think that a maximum principle working for $W^{1,p}$ weak minimizers can not be obtained just with a slight modification of the proof. Beside the lack of continuity, one of the main problems is that the class of $W^{1,p}(M,N)$ maps is defined via $W^{1,p}(M,\mathbb{R}^q)$ using the isometric embedding of N in $\mathbb{R}^q$ provided by Nash theorem. Accordingly one should work on $\mathbb{R}^q$ using the extrinsic expression for the p-harmonicity; see for instance formula (2) in [PV2]. Now, no information on the second fundamental form of the immersion is known. A suitable intrinsic notion of Sobolev class of maps could help to overcome this problem. We are planning to investigate this order of ideas in a future paper.

Proof. Let $w^* = \sup_{\partial M} w$ and, by contradiction, suppose that $w(x_0) > w^*$ for some $x_0 \in \operatorname{int}(M)$. Fix $0 < \varepsilon << 1$ so that $w(x_0) - w^* > 2\varepsilon$. Let $\lambda : \mathbb{R} \to [0,1]$ satisfy $\lambda' \geq 0$, $\lambda' > 0$ on $(\varepsilon, +\infty)$, $\lambda = 0$ on $(-\infty, \varepsilon]$. Define the vector field

$$Z = |du|^{p-2}\,\lambda(w - w^*)\nabla w$$

and note that $\operatorname{supp} Z \subset \operatorname{int}(M)$. Direct computations show that

$$\begin{aligned}\operatorname{div} Z &= \lambda' \circ (w - w^*)\,|du|^{p-2}\,|\nabla w|^2\\ &\quad + \lambda \circ (w - w^*)\,\operatorname{tr}\operatorname{Hess}(f)\left(|du|^{p-2}\,du, du\right)\\ &\quad + \lambda \circ (w - w^*) df\,(\Delta_p u)\\ &\geq |\nabla w|^2\,|du|^{p-2}\,\lambda' \circ (w - w^*),\end{aligned}$$

and applying the divergence theorem we get

$$0 \leq \int_M |\nabla w|^2\,|du|^{p-2}\,\lambda' \circ (w - w^*) \leq \int_M \operatorname{div} Z = 0.$$

This proves that

$$|\nabla w|^2\,|du|^{p-2} = 0 \text{ on } M_\varepsilon, \tag{14}$$

where we have denoted with M_ϵ the connected component containing x_0 of the open set

$$\{x \in M : w - w^* - \varepsilon > 0\}.$$

Since, by (14), $dw = df\,(du) = 0$ where $du \neq 0$ and $dw = df\,(du) = 0$ where $du = 0$, it follows that w is constant on M_ε and this easily gives the desired contradiction. □

6. Proof of the main results

In this last Section we put all the previous ingredients together to get a proof of Theorems B and Theorem C.

The boundary datum f has image confined in a ball $B^N_{R_0}(0)$ of N^n. Using Theorem 2.2 (or Theorem 2.3), we glue $B^N_{R_0}(0)$ to the exterior of a large ball in the hyperbolic spaceform $\mathbb{H}^n_{-k^2}$ of sufficiently negative curvature

$-k^2 << -1$, say $\mathbb{H}^n_{-k^2} \backslash \mathbb{B}_{R_1}(0)$, $R_1 >> R_0$, thus obtaining a new Cartan-Hadamard model manifold (N', h'). On the other hand, by Proposition 4.2, $\mathbb{H}^n_{-k^2}$ has compact quotients with arbitrarily large injectivity radii. Accordingly, we can choose a discrete subgroup Γ of isometries acting freely and co-compactly on $\mathbb{H}^n_{-k^2}$ in such a way that $\mathbb{B}_{R_1}(0)$ is contained in a relatively compact, fundamental domain of the action, say $\mathbb{B}_{R_1}(0) \subset\subset \Omega$. Making use of Γ we extend the deformed metric of $\bar{\Omega}$ periodically thus obtaining a new Riemannian manifold N'' diffeomorphic to $\mathbb{H}^m_{-k^2}$. More precisely, the metric h'' of N'' is defined by setting

$$h''_{\gamma \cdot p} = \left(\gamma^{-1}\right)^*_{\gamma \cdot p} h'_p.$$

Since h' is hyperbolic in a neighborhood of $\partial\Omega$, the definition of h'' is well posed. Moreover, (N'', h'') has non-positive curvature, hence it is Cartan-Hadamard, and, by construction, Γ acts freely and co-compactly by isometries on N''. In particular, each copy of Ω contains an isometric image of $B^N_{R_0}(0)$. Now, we take the quotient manifold N''/Γ which is compact and covered by N'' via the quotient projection $P : N'' \to N''/\Gamma$. By construction, the original datum f well defines $f'' = f : M \to N''$. Applying Theorem A we get a unique solution $u'' \in C^0(M, N'') \cap C^{1,\alpha}(\mathrm{int}(M), N'')$ to the Dirichlet problem

$$\begin{cases} \Delta_p u'' = 0 & \text{on } M \\ u'' = f'' & \text{on } \partial M. \end{cases}$$

To complete the argument, it remains to show that, actually, u'' gives rise to a solution of the original problem. This clearly follows if we are able to show that its image is confined in $B^N_{R_0}(0) \subset N''$. To prove that this is the case, we recall that N'' is Cartan-Hadamard and, therefore, the function $d^2_{N''}(y, 0)$ is smooth and strictly convex. By means of Theorem 5.1, we deduce that $d^2_{N''}(u'', 0)$ achieves its maximum on ∂M. To conclude, it suffices to recall that $f(M) \subset B^N_{R_0}(0)$ and to use the equality $u'' = f$ on ∂M.

Remark 6.1. In view of known results in the harmonic case, [HKW1, HKW2, HKW3], it is an interesting problem to extend the conclusion of Theorem B to the case where N is a general Cartan-Hadamard manifold or it is replaced by a regular ball in a complete manifold. Apparently, the above strategy cannot be readily adapted to these situations. One of the main obstructions is that, despite of the use of Theorem 3.1 to obtain N' supporting a strictly convex exhaustion function, we are not able to show that such a nice function can be constructed also on N''. This latter is the complete, simply connected manifold which gives rise to a compact quotient and, therefore, that enables us to apply the results in [PV2]. Once this problem is solved, we should also relate the convex function on N'' to the size of the unperturbed ball $B^N_{R_0}$ in such a way that, applying the maximum principle, we can conclude that the lifted solution is confined in $B^N_{R_0}$.

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