

Global Comparison Principles for the p -Laplace Operator on Riemannian Manifolds

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Abstract We prove global comparison results for the p -Laplacian on a p -parabolic manifold. These involve both real-valued and vector-valued maps with finite p -energy. Further L^q comparison principles in the non-parabolic setting are also discussed.

Keywords Nonlinear comparisons · Finite p -energy · p -parabolicity

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1 Introduction and Statement of the Main Results

Let $(M, \langle \cdot, \cdot \rangle)$ be a connected, m -dimensional, complete Riemannian manifold and let $p > 1$. Recall that the p -Laplacian of a real valued function $u : M \rightarrow \mathbb{R}$ is defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. Following the classical terminology, see e.g. [3], a function $u \in W_{\text{loc}}^{1,p}(M)$ is said to be p -subsolution if $\Delta_p u \geq 0$ weakly on M . In case any

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bounded above p -subsolution is necessarily constant we say that the manifold M is p -parabolic. It is known that p -parabolicity is related to volume growth properties of the underlying manifold. Accordingly, M is p -parabolic provided, for some $x \in M$,

$$\left(\frac{r}{\text{vol}_m B_r(x)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty), \tag{1}$$

where $B_r(x)$ denotes the metric ball centered at x , of radius $r > 0$, and vol_m is the m -dimensional Hausdorff measure. Thus, for instance, the standard Euclidean space \mathbb{R}^m is p -parabolic if $m \leq p$. Condition (1) is quite natural in that it shares the quasi-isometry invariance of p -parabolicity. Moreover, it turns out that there are geometric situations where Eq. 1 is also necessary for M to be p -parabolic; see [5, 7] and references therein. On the other hand, it was established in [16, 18] and [6] that the most general volume growth condition ensuring p -parabolicity is that, for some $x \in M$,

$$\left(\frac{1}{\text{vol}_{m-1} \partial B_r(x)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty).$$

Now, suppose that M is p -parabolic, with $p \geq 2$. It is known, [13], that a smooth p -subsolution $u : M \rightarrow \mathbb{R}$ with finite p -energy $|\nabla u| \in L^p(M)$ must be constant. We shall show that this is nothing but a very special case of a genuine comparison principle for the p -Laplace operator.

Recall that, given a function $f \in L^1_{\text{loc}}(M)$ and a vector field $X \in L^1_{\text{loc}}(M)$, we say that $\text{div } X \geq f$ weakly (or in the sense of distributions) on M if, for all non-negative, compactly supported, smooth test functions φ , $0 \leq \varphi \in C^\infty_c(M)$,

$$(\text{div } X, \varphi) := - \int \langle X, \nabla \varphi \rangle \geq \int f \varphi. \tag{2}$$

In particular, if $X = |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v$ for some real-valued functions $u, v \in W^{1,p}_{\text{loc}}(M)$ and $f \equiv 0$, we have that the weak inequality $\Delta_p u \geq \Delta_p v$ means

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \leq \int \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle, \tag{3}$$

for all $0 \leq \varphi \in C^\infty_c(M)$. Note that, by standard density results and by dominated convergence, it is equivalent to require the validity of Eqs. 2 and 3 for all $0 \leq \varphi \in W^{1,p}_c(M)$ if $|\nabla u|, |\nabla v| \in L^p(M)$. Above $W^{1,p}_{\text{loc}}(M)$ stands for the (local) Sobolev space of all functions $u \in L^p_{\text{loc}}(M)$ whose weak (distributional) gradients also belong to $L^p_{\text{loc}}(M)$. Furthermore, $W^{1,p}_c(M)$ is the closure of $C^\infty_c(M)$ in $W^{1,p}(M)$.

Theorem 1 *Let $(M, \langle \cdot, \cdot \rangle)$ be a connected, possibly incomplete, p -parabolic Riemannian manifold, with $p > 1$. Assume that $u, v \in W^{1,p}_{\text{loc}}(M) \cap C^0(M)$ satisfy*

$$\Delta_p u \geq \Delta_p v \text{ weakly on } M,$$

and

$$|\nabla u|, |\nabla v| \in L^p(M).$$

Then $u - v$ is constant.

Simple examples show that both the p -parabolicity of M and the L^p -integrability of $|\nabla u|$ or $|\nabla v|$ are needed above. Indeed, let M be, for instance, the open unit ball in \mathbb{R}^m , u a constant function, and v a non-constant p -harmonic function in M (i.e. a continuous weak solution to $\Delta_p v = 0$), with $|\nabla v| \in L^p(M)$. Then M is non- p -parabolic for all $p > 1$ and the conclusion of Theorem 1 clearly fails. On the other hand, let M be the infinite cylinder $\mathbb{R} \times \mathbb{S}^{m-1}$ equipped with the product metric $ds^2 = dr^2 + d\vartheta^2$, where $d\vartheta^2$ is the standard metric of the sphere \mathbb{S}^{m-1} . Furthermore, let u be a constant function and $v(t, \vartheta) = t$. Now M is p -parabolic for all $p > 1$, u and v are p -harmonic in M , but the conclusion of Theorem 1 again fails.

To prove Theorem 1 we will introduce an inequality for the p -Laplacian which resembles a well known inequality for the mean curvature operator. A basic use of this inequality will enable us to get also the next result in the spirit of [12].

Theorem 2 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold. Let $u, v \in C^\infty(M)$ be such that*

$$\Delta_p u \geq \Delta_p v \text{ on } M$$

for some $p \geq 2$. Suppose there exist $q \geq 1$ and $s > p$ such that

$$\int^{+\infty} f_{p,q,s}(r) dr = +\infty, \tag{4}$$

where we have set

$$f_{p,q,s}(r) = \left(\int_{\partial B_r(o)} |u - v|^{q+\frac{1}{s-1}} (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}} \right)^{1-s}, \tag{5}$$

for some $o \in M$. Then either $u - v$ is constant or $u \leq v$ on M .

Condition (4) could appear a little bit hard to decode due the presence of many parameters. This condition will become more transparent in Remark 13 at the end of Section 4 where a few special cases of particular interest are discussed.

Besides real-valued functions one is naturally led to consider manifold-valued maps. Several topological questions are related to the p -Laplacian of maps; [15, 19]. Recall that the p -Laplacian (or the p -tension field) of a map $u : M \rightarrow N$ between Riemannian manifolds is defined by

$$\Delta_p u = \operatorname{div} \left(|du|^{p-2} du \right).$$

Here, $du \in T^*M \otimes u^{-1}TN$ denotes the differential of u and the bundle $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product $\langle \cdot, \cdot \rangle$. Moreover, $-\operatorname{div}$ stands for the formal adjoint of the exterior differential d with respect to the standard L^2 inner product on vector-valued 1-forms. Say that u is p -harmonic if $\Delta_p u = 0$. In [17], Schoen and Yau prove a general comparison principle for homotopic (2-)harmonic maps with finite (2-)energy into non-positively curved targets. They assume that the complete, non-compact manifold M has finite volume but the request that M is (2-)parabolic suffices, [13]. In this direction, comparisons for homotopic p -harmonic maps with finite p -energy into non-positively curved manifolds are far from being completely understood. Some progress in the special situation of a single map homotopic to a constant has been made in [13]. In this note, we focus our

attention on the case $N = \mathbb{R}^n$. According to [13], it is clear that, if M is p -parabolic, then every p -harmonic map $u : M \rightarrow \mathbb{R}^n$ with finite p -energy $|du| \in L^p(M)$ must be constant. However, using the very special structure of \mathbb{R}^n , we are able to extend this conclusion, thus establishing a comparison principle for maps $u, v : M \rightarrow \mathbb{R}^n$ having the same p -Laplacian. In some sense, this can be considered as a further step towards the comprehension of the general comparison problem alluded to above.

Theorem 3 *Suppose that (M, \langle, \rangle) is a connected, possibly incomplete, p -parabolic Riemannian manifold, with $p \geq 2$. Let $u, v : M \rightarrow \mathbb{R}^n$ be smooth maps satisfying*

$$\Delta_p u = \Delta_p v \text{ on } M, \quad (6)$$

and

$$|du|, |dv| \in L^p(M).$$

Then $u - v$ is constant.

The paper is organized as follows: in Section 2 we collect some of the key tools needed in the proofs of the main theorems. Section 3 is devoted to a proof of the finite-energy comparisons stated in Theorems 1 and 3. In the final Section 4 we give a prove of Theorem 2.

2 Main Tools

In the proofs of Theorems 1 and 3 we will use two main ingredients: (a) a version for the p -Laplacian of a classical inequality for the mean-curvature operator, which will be also used in a final section to prove Theorem 2; (b) a global form of the divergence theorem in non-compact settings which inspires to a p -parabolicity criterion involving vector fields.

2.1 A key inequality

The following basic inequality was discovered by Lindqvist, [9].

Lemma 4 *Let (V, \langle, \rangle) be a finite dimensional, real vector space endowed with a positive definite scalar product and let $p > 1$. Then, for every $x, y \in V$ it holds*

$$|x|^p + (p - 1)|y|^p - p|y|^{p-2} \langle x, y \rangle \geq C(p)\Psi(x, y),$$

where

$$\Psi(x, y) := \begin{cases} |x - y|^p & p \geq 2 \\ \frac{|x - y|^2}{(|x| + |y|)^{2-p}} & 1 < p < 2, \end{cases}$$

and $C(p)$ is a positive constant depending only on p .

As a consequence, we deduce the validity of the next

Corollary 5 *In the above assumptions, for every $x, y \in V$, it holds*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq 2C(p)\Psi(x, y). \tag{7}$$

Proof We start computing

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle = |x|^p + |y|^p - \langle x, y \rangle (|x|^{p-2} + |y|^{p-2}).$$

On the other hand, applying twice Lindqvist inequality with the role of x and y interchanged we get

$$p(|x|^p + |y|^p) \geq p(|x|^{p-2} + |y|^{p-2}) \langle x, y \rangle + 2C(p)\Psi(x, y).$$

Inserting into the above completes the proof. □

Remark 6 Inequality (7) can be considered as a version for the p -Laplacian of the classical Miklyukov-Hwang-Collin-Krust inequality; [1, 8, 11]. This latter states that, for every $x, y \in V$,

$$\left\langle \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}, x - y \right\rangle \geq \frac{\sqrt{1+|x|^2} + \sqrt{1+|y|^2}}{2} \left| \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}} \right|^2,$$

equality holding if and only if $x = y$. This analogy suggests the validity of global comparison results, without any p -parabolicity assumption, in the spirit of [12], as exemplified by Theorem 2. See Section 4.

2.2 p -parabolicity and Related Properties

As we mentioned in the introduction, a manifold M is p -parabolic if a Liouville type property holds for p -subsolutions that are bounded above. It is well known that this is just one of the several equivalent definitions of p -parabolicity; see [4]. For instance, and in view of future purposes, we recall the next

Theorem 7 *The manifold M is p -parabolic if and only if the (relative) p -capacity of every compact set K vanishes. This means that*

$$\inf \int_M |\nabla \varphi|^p = 0$$

where the infimum is taken over all compactly supported smooth functions φ satisfying $\varphi = 1$ on K .

A further very useful characterization of (non-) p -parabolicity involves special vector fields on the underlying manifold. It goes under the name of Kelvin-Nevanlinna-Royden criterion. In the linear setting $p = 2$ it was proved in a paper

by Lyons and Sullivan [10]. See also Theorem 7.27 in [14]. The following non-linear extension is due to Gol'dshtein and Troyanov [2].

Theorem 8 *The manifold M is not p -parabolic if and only if there exists a vector field X on M such that:*

- (a) $|X| \in L^{\frac{p}{p-1}}(M)$
- (b) $\operatorname{div} X \in L^1_{loc}(M)$ and $\min(\operatorname{div} X, 0) = (\operatorname{div} X)_- \in L^1(M)$
- (c) $0 < \int_M \operatorname{div} X \leq +\infty$.

Accordingly, if M is p -parabolic and X is a vector field satisfying (a) $|X| \in L^{\frac{p}{p-1}}(M)$, (b) $\operatorname{div} X \in L^1_{loc}(M)$, and (c) $\operatorname{div} X \geq 0$ on M , then we must necessarily conclude that $\operatorname{div} X = 0$ on M . It is worth pointing out that, even if condition (b) is not satisfied, we can obtain a similar conclusion as shown in the next

Proposition 9 *Let $(M, \langle \cdot, \cdot \rangle)$ be a p -parabolic Riemannian manifold, $p > 1$. Let X be a vector field satisfying $|X| \in L^{\frac{p}{p-1}}(M)$ and*

$$\operatorname{div} X \geq f \geq 0$$

in the sense of distributions, for some $0 \leq f \in L^1_{loc}(M)$. Then

$$f \equiv 0.$$

Proof Let $\{\Omega_j\}_{j=0}^\infty$ be an increasing sequence of precompact open sets with smooth boundaries such that $\Omega_j \nearrow M$. Let φ_j be the p -equilibrium potential of the condenser $C(\Omega_j, \overline{\Omega_0})$, namely

$$\int_M |\nabla \varphi_j|^p = \min \int_M |\nabla \varphi|^p$$

where the minimum is taken over all smooth φ compactly supported in Ω_j and satisfying $\varphi = 1$ on $\overline{\Omega_0}$. Then, φ_j solves the Dirichlet problem

$$\begin{cases} \Delta_p \varphi_j = 0 & \Omega_j \setminus \overline{\Omega_0} \\ \varphi_j = 1 & \text{on } \overline{\Omega_0} \\ \varphi_j = 0 & \text{on } \partial\Omega_j \end{cases}$$

and we have

$$\begin{aligned} 0 &\leq \int_M \varphi_j f \leq (\operatorname{div} X, \varphi_j) \\ &= - \int_M \langle X, \nabla \varphi_j \rangle \\ &\leq \left(\int_M |X|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_M |\nabla \varphi_j|^p \right)^{\frac{1}{p}}. \end{aligned} \quad (8)$$

Note that, by Theorem 7,

$$\int_M |\nabla \varphi_j|^p \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

which implies that the RHS of Eq. 8 vanishes as $j \rightarrow \infty$. Moreover, by the comparison principle on precompact domains it follows that $0 \leq \varphi_j \leq 1$ is a non-decreasing sequence of functions pointwise converging to some $\varphi > 0$. Hence, taking limits in Eq. 8 and using monotone convergence,

$$0 \leq \int_M \varphi f \leq 0$$

and this latter gives $f \equiv 0$. □

Remark 10 Using an Ahlfors type characterization of p -parabolicity in terms of a boundary maximum principle for p -harmonic functions on generic domains we see that, in fact, $\varphi \equiv 1$.

3 Proofs of the Finite-energy Comparison Principles

We are now in the position to prove the main results.

Proof of Theorem 1 Fix any $x_0 \in M$, let $A = u(x_0) - v(x_0)$ and define Ω_A to be the connected component of the open set

$$\{x \in M : A - 1 < u(x) - v(x) < A + 1\}$$

which contains x_0 . By standard topological arguments, $\Omega_A \neq \emptyset$ is a (connected) open set. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be the piece-wise linear function defined by

$$\alpha(t) = \begin{cases} 0 & t \leq A - 1 \\ (t - A + 1) / 2 & A - 1 \leq t \leq A + 1 \\ 1 & t \geq A + 1. \end{cases}$$

Consider the vector field

$$X = \alpha \circ (u - v) \{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \},$$

and note that, for a suitable constant $C > 0$,

$$|X|_{\frac{p}{p-1}} \leq C (|\nabla u|^p + |\nabla v|^p) \in L^1(M).$$

From now on we abbreviate $\alpha(u - v) = \alpha \circ (u - v)$, $\alpha'(u - v) = \alpha' \circ (u - v)$, etc. Since $\alpha(u - v) \in W_{\text{loc}}^{1,p}(M)$ then, by assumption, for all functions $0 \leq \varphi \in C_c^\infty(M)$ we have

$$\begin{aligned} 0 &\geq \int \langle \nabla(\varphi\alpha(u - v)), |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \rangle \\ &= \int \langle \nabla\varphi, \alpha(u - v) \{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \} \rangle \\ &\quad + \int \varphi \alpha'(u - v) \langle \nabla u - \nabla v, |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \rangle \\ &\geq -(\text{div } X, \varphi) + 2C(p) \int \varphi \alpha'(u - v) \Psi(x, y) \end{aligned}$$

where in the last inequality we have used Corollary 5 and the fact that $\alpha' \geq 0$. Then

$$\text{div } X \geq 2C(p)\alpha'(u - v) \Psi(x, y) \geq 0$$

in the sense of distributions and Proposition 9 yields

$$\alpha'(u - v) |\nabla u - \nabla v| = 0.$$

Since $\alpha'(u - v) \neq 0$ on Ω_A , we deduce

$$u - v \equiv A, \text{ on } \Omega_A.$$

It follows that the open set Ω_A is also closed. Since M is connected we must conclude that $\Omega_A = M$ and $u - v = A$ on M . \square

Remark 11 In the above proof, inequality (7) is not used in its full strength. What we really need is that

$$\langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla u - \nabla v \rangle > 0$$

whenever $\nabla u \neq \nabla v$. According to this observation, the same proof works with minor changes for more general operators such as the \mathcal{A} -Laplacian of [3] or the φ -Laplacian of [16]. In this latter case, $\varphi(t)$ is required to be increasing.

Proof of Theorem 3 We suppose that either u or v is non-constant, for otherwise there's nothing to prove. Fix $q_0 \in M$. Set $C := u(q_0) - v(q_0) \in \mathbb{R}^n$ and introduce the radial function $r : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as $r(x) = |x - C|$. For $T > 0$, consider the piecewise differentiable vector field X_T on M defined as

$$X_T(x) := \left[dh_T|_{(u-v)(x)} \circ (|du(x)|^{p-2} du(x) - |dv(x)|^{p-2} dv(x)) \right]^\sharp, \quad x \in M,$$

where $h_T \in C^1(\mathbb{R}^n, \mathbb{R})$ is the function

$$h_T(x) := \begin{cases} \frac{r^2(x)}{2} & \text{if } r(x) < T \\ Tr(x) - \frac{T^2}{2} & \text{if } r(x) \geq T \end{cases}$$

and \sharp denotes the isomorphism defined by using the Riemannian metric as $\langle \omega^\sharp, V \rangle = \omega(V)$ for all differential 1-forms ω and vector fields V . We observe that $h_T \in C^2$ where $r(x) \neq T$ and that X_T is well defined since there exists a canonical identification

$$T_{(u-v)(q)}\mathbb{R}^n \cong T_{u(q)}\mathbb{R}^n \cong T_{v(q)}\mathbb{R}^n \cong \mathbb{R}^n.$$

We also observe that, by Sard theorem, for i.e. $T > 0$, the level set $\{|u - v - C| = T\}$ is a smooth (possibly empty) hypersurface, hence a set of measure zero. Thus, the vector field X_T is weakly differentiable and, for i.e. $T > 0$, the weak divergence of X_T is given by

$$\begin{aligned} \operatorname{div} X_T &= d\left(\frac{r^2}{2}\right)|_{(u-v)} \circ (\Delta_p u - \Delta_p v) \\ &\quad + {}^M \operatorname{tr} \left(\operatorname{Hess} \left(\frac{r^2}{2} \right) |_{(u-v)} (du - dv, |du|^{p-2} du - |dv|^{p-2} dv) \right) \end{aligned}$$

if $r(x) < T$ and

$$\begin{aligned} \operatorname{div} X_T &= d(Tr)|_{(u-v)} \circ (\Delta_p u - \Delta_p v) \\ &\quad + {}^M \operatorname{tr} (\operatorname{Hess}(Tr)|_{(u-v)} (du - dv, |du|^{p-2} du - |dv|^{p-2} dv)) \end{aligned}$$

if $r(x) \geq T$. In both cases the first term on the RHS vanishes by assumption. Moreover, by standard computations, we have $\operatorname{Hess}(r) = r^{-1}(\langle \cdot, \cdot \rangle_{\mathbb{R}^n} - dr \otimes dr)$ on $\mathbb{R}^n \setminus \{C\}$. Thus,

$$\begin{aligned} \operatorname{Hess} \left(\frac{r^2}{2} \right) &= dr \otimes dr + r \operatorname{Hess}(r) = \langle \cdot, \cdot \rangle_{\mathbb{R}^n} \quad \text{if } r(x) < T, \\ \operatorname{Hess}(Tr) &= T \operatorname{Hess}(r) = \frac{T}{r} (\langle \cdot, \cdot \rangle_{\mathbb{R}^n} - dr \otimes dr) \quad \text{if } r(x) \geq T. \end{aligned}$$

As a consequence, for $q \in M$ such that $r((u - v)(q)) < T$, by Corollary 5 we get

$$\operatorname{div} X_T = \langle du - dv, |du|^{p-2} du - |dv|^{p-2} dv \rangle \geq 2C(p)|du - dv|^p, \tag{9}$$

while, for $q \in M$ such that $r((u - v)(q)) \geq T$, it holds

$$\begin{aligned} \operatorname{div} X_T &= \frac{T}{r(u - v)} \langle du - dv, |du|^{p-2} du - |dv|^{p-2} dv \rangle \\ &\quad - \frac{T}{r(u - v)} \langle dr|_{(u-v)}(du - dv), dr|_{(u-v)}(|du|^{p-2} du - |dv|^{p-2} dv) \rangle \\ &\geq \frac{T}{r(u - v)} 2C(p)|du - dv|^p - (|du| + |dv|)(|du|^{p-1} + |dv|^{p-1}) \\ &\geq \frac{T}{r(u - v)} 2C(p)|du - dv|^p - (|du|^p + |dv|^p + |du|^{p-1}|dv| + |dv|^{p-1}|du|) \\ &\geq \frac{T}{r(u - v)} 2C(p)|du - dv|^p - 2(|du|^p + |dv|^p), \end{aligned} \tag{10}$$

where we have used again Corollary 5 for the first term and Cauchy-Schwarz inequality, Young’s inequality and the facts that $|dr| = 1$ and $r(u - v) \geq T$ for the second one. Let us now compute the $L^{\frac{p}{p-1}}$ -norm of X_T . Since

$$||du|^{p-2}du - |dv|^{p-2}dv|^{\frac{p}{p-1}} \leq (|du|^{p-1} + |dv|^{p-1})^{\frac{p}{p-1}} \leq 2^{\frac{1}{p-1}} (|du|^p + |dv|^p),$$

we have

$$\begin{aligned} \int_{\{|u-v-C|<T\}} |X_T|^{\frac{p}{p-1}} &\leq \int_{\{|u-v-C|<T\}} |u - v - C|^{\frac{p}{p-1}} ||du|^{p-2}du - |dv|^{p-2}dv|^{\frac{p}{p-1}} \\ &\leq T^{\frac{p}{p-1}} 2^{\frac{1}{p-1}} (\|du\|_p^p + \|dv\|_p^p) < +\infty \end{aligned}$$

and

$$\begin{aligned} \int_{\{|u-v-C|>T\}} |X_T|^{\frac{p}{p-1}} &\leq \int_{\{|u-v-C|>T\}} T^{\frac{p}{p-1}} ||du|^{p-2}du - |dv|^{p-2}dv|^{\frac{p}{p-1}} \\ &\leq T^{\frac{p}{p-1}} 2^{\frac{1}{p-1}} (\|du\|_p^p + \|dv\|_p^p) < +\infty. \end{aligned}$$

Hence X_T is a weakly differentiable vector field with $|X_T| \in L^{\frac{p}{p-1}}(M)$ and $\operatorname{div} X_T \in L^1_{\text{loc}}(M)$. To apply Theorem 8, it remains to show that $(\operatorname{div} X_T)_- \in L^1(M)$. By inequalities (9) and (10), we deduce that

$$\int_M |(\operatorname{div} X_T)_-| \leq 2 \int_{\{|u-v-C|>T\}} (|du|^p + |dv|^p) \leq 2(\|du\|_p^p + \|dv\|_p^p) < +\infty. \tag{11}$$

Then, the assumptions of Theorem 8 are satisfied and we get, for i.e. $T > 0$,

$$\int_M \operatorname{div} X_T \leq 0.$$

According to Eq. 11 we now choose a sequence $T_n \nearrow +\infty$ such that

$$\int_M |(\operatorname{div} X_{T_n})_-| \leq 2 \int_{\{|u-v-C|>T_n\}} (|du|^p + |dv|^p) < \frac{1}{n}.$$

As a consequence,

$$\begin{aligned} \int_{\{|u-v-C|<T_n\}} 2C(p)|du - dv|^p &\leq \int_{\{|u-v-C|<T_n\}} (\operatorname{div} X_{T_n})_+ \\ &\leq \int_M (\operatorname{div} X_{T_n})_+ \\ &\leq - \int_M (\operatorname{div} X_{T_n})_- < \frac{1}{n}. \end{aligned} \tag{12}$$

Therefore, letting n go to $+\infty$, we obtain

$$\int_M C(p)|d(u - v)|^p = 0,$$

that is, $u - v \equiv u(q_0) - v(q_0) = C$ on M . □

4 Further Comparison Results Without Parabolicity

In this last section we give a proof of Theorem 2. Note that the techniques developed in [12] can be used to conclude further (e.g. L^∞) comparison results. We shall need the following lemma

Lemma 12 *Let $p \geq 2$. Then, for every $x, y \in \mathbb{R}^n$, it holds*

$$||x|^{p-2}x - |y|^{p-2}y| \leq (p - 1)(|x| + |y|)^{p-2}|x - y|.$$

Proof Set $E(x) := |x|^{p-2}x$. We start by computing

$$\begin{aligned} \left| \frac{d}{dt} E(tx + (1 - t)y) \right| &\leq (p - 1)|tx + (1 - t)y|^{p-2}|x - y| \\ &\leq (p - 1)(|x| + |y|)^{p-2}|x - y|, \end{aligned}$$

from which we obtain

$$\begin{aligned} |E(x) - E(y)| &= \left| \int_0^1 \frac{d}{dt} E(tx + (1 - t)y) dt \right| \\ &\leq \int_0^1 \left| \frac{d}{dt} E(tx + (1 - t)y) \right| dt \\ &\leq (p - 1)(|x| + |y|)^{p-2}|x - y|. \end{aligned}$$

□

Proof of Theorem 2 First of all, for the ease of notation, we set

$$E(\xi) := |\xi|^{p-2}\xi, \xi \in TM.$$

Suppose that $u - v$ is not constant and, by contradiction, assume that there exists a point $x_0 \in M$ such that $u(x_0) > v(x_0)$. Fix a real number $0 < \epsilon < (u(x_0) - v(x_0))/2$ and define Ω_ϵ to be the connected component of the open set $\{x \in M : u(x) - v(x) > \epsilon\}$ which contains x_0 . Note that, necessarily, $u - v$ is not constant on Ω_ϵ . Indeed, otherwise, by standard topological arguments we would have $\Omega_\epsilon = M$ and $u - v$ would be constant on all of M . We choose a smooth, non-decreasing function λ such that $\lambda(t) = 0$ for every $t < 2\epsilon$ and $0 < \lambda(t) \leq 1$ for every $t > 2\epsilon$ and we define the vector field

$$X := \lambda(u - v)(u - v)^q (E(\nabla u) - E(\nabla v)).$$

We write B_R for $B_R(o)$ and $\partial/\partial r$ for the radial vector field centered at o . Applying the divergence theorem, Lemma 12 and Hölder inequality, we get

$$\begin{aligned} \int_{B_R \cap \Omega_\epsilon} \operatorname{div} X &= \int_{\partial B_R \cap \Omega_\epsilon} \left\langle X, \frac{\partial}{\partial r} \right\rangle \leq \int_{\partial B_R \cap \Omega_\epsilon} |E(\nabla u) - E(\nabla v)| \lambda(u-v)(u-v)^q \\ &\leq (p-1) \int_{\partial B_R \cap \Omega_\epsilon} \lambda(u-v)(|\nabla u| + |\nabla v|)^{p-2} |\nabla u - \nabla v| (u-v)^q \\ &\leq (p-1) \left(\int_{\partial B_R \cap \Omega_\epsilon} F(u, v) \right)^{\frac{1}{s}} \\ &\quad \times \left(\int_{\partial B_R \cap \Omega_\epsilon} \lambda(u-v)(|\nabla u| + |\nabla v|)^{\frac{(p-2)s}{s-1}} |\nabla u - \nabla v|^{(1-\frac{p}{s})\frac{s}{s-1}} (u-v)^{\frac{sq-q+1}{s-1}} \right)^{\frac{s-1}{s}} \\ &\leq (p-1) \left(\int_{\partial B_R \cap \Omega_\epsilon} F(u, v) \right)^{\frac{1}{s}} \left(\int_{\partial B_R} |u-v|^{q+\frac{1}{s-1}} (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}} \right)^{\frac{s-1}{s}}, \end{aligned}$$

where

$$F(u, v) = \lambda(u-v)|\nabla u - \nabla v|^p (u-v)^{q-1}$$

and, we recall, $s > p$. On the other hand, computing the divergence of X we obtain

$$\begin{aligned} \int_{B_R \cap \Omega_\epsilon} \operatorname{div} X &= \int_{B_R \cap \Omega_\epsilon} \lambda'(u-v)(u-v)^q \langle E(\nabla u) - E(\nabla v), \nabla u - \nabla v \rangle \\ &\quad + q \int_{B_R \cap \Omega_\epsilon} (u-v)^{q-1} \lambda(u-v) \langle E(\nabla u) - E(\nabla v), \nabla u - \nabla v \rangle \\ &\quad + \int_{B_R \cap \Omega_\epsilon} (\Delta_p u - \Delta_p v) \lambda(u-v)(u-v)^q \\ &\geq 2qC(p) \int_{B_R \cap \Omega_\epsilon} F(u, v), \end{aligned}$$

where, in the last inequality, we have used Corollary 5. It follows that

$$H(R)^s \leq C' \xi(R) H'(R), \tag{13}$$

where we have defined

$$\begin{aligned} H(R) &:= \int_{B_R \cap \Omega_\epsilon} F(u, v) \geq 0; \\ \xi(R) &:= \left(\int_{\partial B_R} |u-v|^{q+\frac{1}{s-1}} (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}} \right)^{s-1} \\ C' &:= (p-1)^s [2qC(p)]^{-s}. \end{aligned}$$

Choose $r_1 \gg 1$ such that $F(u, v)$ does not vanish identically on $B_{r_1} \cap \Omega_\epsilon$. According to Eq. 13 we have $\xi(R), H(R) > 0$, for every $R \geq r_1$. Therefore, we can integrate Eq. 13 on $[r_1, r_2]$ to obtain

$$\begin{aligned} \left(\frac{C'}{s-1}\right) \frac{1}{H(r_1)^{s-1}} &\geq \left(\frac{C'}{s-1}\right) (-H(r_2)^{1-s} + H(r_1)^{1-s}) \\ &\geq \int_{r_1}^{r_2} \frac{dt}{\xi(t)}. \end{aligned} \tag{14}$$

Letting $r_2 \rightarrow \infty$, the RHS of Eq. 14 goes to infinity by assumption, and this force $H(r_1) = 0$ for all r_1 . Hence

$$\nabla(u - v) \equiv 0 \text{ on } \Omega_\epsilon$$

proving that $u - v$ is constant on Ω_ϵ . Contradiction. □

Remark 13 Note that the assumptions of Theorem 2 are trivially met if either $(u - v)$ or $(|\nabla u| + |\nabla v|)$ has compact support. More importantly, applying Hölder and reverse Hölder inequalities, we can see that condition (4) in Theorem 2 is implied by the stronger assumption

$$\left(\int^R \| |u - v|^{q+\frac{1}{s-1}} \|_{t, \partial B_r}^{-\frac{s-1}{z}} dr\right)^z \left(\int^R \| (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}} \|_{\frac{t}{t-1}, \partial B_r}^{\frac{s-1}{t-1}} dr\right)^{1-z} \nearrow \infty,$$

as $R \rightarrow \infty$, for some $t \in [1, +\infty]$ and $z \in (-\infty, 0) \cup (1, +\infty)$. Here $\|f\|_{t, \Omega}$ denotes the L^t norm of f on Ω . In particular we obtain that Theorem 2 holds if we replace Eq. 4 with either of the following set of assumptions:

- 4.i) $|\nabla u|, |\nabla v| \in L^\infty(M)$ and $\left[\int_{\partial B_r} |u - v|^{q+\frac{1}{s-1}}\right]^{1-s} \notin L^1(+\infty)$ for some $q \geq 1$ and $s > p$;
- 4.ii) $|u - v| \in L^\infty(M)$ and $\left[\int_{\partial B_r} (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}}\right]^{1-s} \notin L^1(+\infty)$ for some $s > p$;
- 4.iii) $|\nabla u|, |\nabla v| \in L^{(p-\frac{s}{s-1})t}(M)$, for some $s > p$ and $t > 1$, and

$$\left[\int_{\partial B_r} |u - v|^{(q+\frac{1}{s-1})\frac{t}{t-1}}\right]^{\frac{(1-s)(t-1)}{s+t-1}} \notin L^1(+\infty)$$

for some $q \geq 1$;

- 4.iv) $|u - v| \in L^{(q+\frac{1}{s-1})t}(M)$, for some $s > p, q \geq 1$ and $t > 1$, and

$$\left[\int_{\partial B_r} (|\nabla u| + |\nabla v|)^{(p-\frac{s}{s-1})\frac{t}{t-1}}\right]^{\frac{(1-s)(t-1)}{s+t-1}} \notin L^1(+\infty).$$

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