THE LICHNEROWICZ EQUATION IN THE CLOSED CASE OF THE EINSTEIN-MAXWELL THEORY

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Abstract. We investigate existence of a solution and stability issues for the Einstein-scalar field Lichnerowicz equation in closed 3-manifolds in the framework of the Einstein-Maxwell theory. The results we obtain provide a complete picture for both the questions of existence and stability.

The fundamental equations of General Relativity are the Einstein equations which link the curvature of the spacetime to its matter content. The equations, in the presence of a cosmological constant, are written as

$$R_{\mu\nu} - \frac{1}{2} S g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \ ,$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, S is the scalar curvature, $g_{\mu\nu}$ is the Lorentzian metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light, and $T_{\mu\nu}$ is the stress-energy tensor. In the Einstein-Maxwell theory, $T_{\mu\nu}$ is the electromagnetic stress-energy tensor. Applying the conformal method of Lichnerowicz, and if we forget about the physical constant $8\pi G/c^4$, solutions to the constraints equations are obtained by solving the system of unknowns (u, W), a positive function and a vector field, consisting of the Lichnerowicz equation

$$\Delta_g u + \frac{1}{8} S_g u = \frac{1}{4}\left(\Lambda - \frac{1}{3}\tau^2\right) u^5 + \frac{|\sigma + DW|^2}{8u^7} + \frac{|E|^2 + |B|^2}{8u^3} \tag{0.1}$$

and the momentum constraint equation

$$\mathrm{div}_g(DW) = \frac{2}{3} u^6 \nabla\tau + E \times B \ , \tag{0.2}$$

where (M, g) is a smooth closed 3-dimensional Riemannian manifold, S_g is the scalar curvature of g, σ is a free 2-tensor field in M, τ is a free function in M representing the mean curvature of the spacelike hypersurface M, D is the conformal Killing operator, and E, B are vector fields representing the electric and magnetic fields. When τ is constant, we see that the two equations (0.1) and (0.2) in the system are actually independent one of another. We assume here that τ is constant and consider that either $W = W(E, B)$ is given out of any physical considerations, or that W is the solution of (0.2). In the second case we need to assume that (M, g) does not possess KIDs (this is somehow a generic situation, see Beig, Chrusciel and Schoen [3]) or that $E \times B$ is L^2-orthogonal to the subspace of conformal Killing fields. Then W is C^1-controlled as soon as we control E and B in the C^0-topology (see Maxwell [15]). We let $\theta > 0$ be constant and consider $(E_\theta, B_\theta) = \theta(E, B)$ so that θ is the coupling constant which measures the strength of the interaction. To

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be coherent with (0.2), there should hold that $W(E_\theta, B_\theta) = \theta^2 W(E, B)$. We let $W_\theta = \theta^2 W(E, B)$, and for homogeneity reasons, we let also $\sigma_\theta = \theta^2\sigma$. We define

$$K(\Lambda, \tau) = \Lambda - \frac{1}{3}\tau^2 , \tag{0.3}$$

and let $Y(M, g)$ be the Yamabe invariant of (M, g) given by

$$Y(M, g) = \frac{1}{8} \inf_{\tilde{g}\in[g]} V_{\tilde{g}}^{-1/3} \int_M S_{\tilde{g}} dv_{\tilde{g}} , \tag{0.4}$$

where $[g]$ is the conformal class of g, and $V_{\tilde{g}}$ is the volume of M with respect to $\tilde{g}$. We consider the Lichnerowicz equation

$$\Delta_g u + \frac{1}{8} S_g u = \frac{1}{4} K(\Lambda, \tau) u^5 + \frac{|\sigma_\theta + DW_\theta|^2}{8u^7} + \frac{|E_\theta|^2 + |B_\theta|^2}{8u^3} , \tag{0.5}$$

and first ask the question of the existence of a solution to this equation. Existence results in the purely Einstein theory in closed manifolds goes back to Isenberg [12]. We refer also to Choquet-Bruhat, Isenberg and Pollack [5], Chrusciel, Galloway and Pollack [6], Hebey, Pacard and Pollack [11], and Maxwell [14]. Our answer to the question of existence in the case of (0.5) is as follows.

Theorem 0.1. *Let (M, g) be a smooth closed 3-manifold, Λ, τ be constants, σ be a smooth $(2, 0)$-tensor field, $(E, B) \not\equiv (0, 0)$ be a nontrivial electromagnetic field, and $\theta > 0$ be the coupling constant. Let $K(\Lambda, \tau)$ be as in (0.3), and $Y(M, g)$ be as in (0.4). Then,*
(1) assuming $Y(M, g) \le 0$, (0.5) possesses a solution if and only if $K(\Lambda, \tau) < 0$,
(2) assuming $Y(M, g) > 0$ and $K(\Lambda, \tau) \le 0$, (0.5) always possesses a solution,
(3) assuming $Y(M, g) > 0$ and $K(\Lambda, \tau) > 0$, there exists $\theta_\star > 0$ such that (0.5) has a solution when $\theta < \theta_\star$, and no solution when $\theta > \theta_\star$.

As a remark, θ plays no role in (1) and (2) in the above theorem. It can be taken to be $\theta = 1$. We assume from now on that $W = W(E, B)$ is controlled in the C^1-topology when E and B are controlled in the C^0-topology in the sense that for any (E, B),

$$W(E', B') \to W(E, B) \text{ in } C^1 \text{ if } (E', B') \to (E, B) \text{ in } C^0 . \tag{0.6}$$

As already mentionned, this is atomatically the case if W is obtained as a solution of the momentum equation (0.2). We define a perturbation of (0.1) to be any sequence of equations which we can write as

$$\Delta_g u + \frac{1}{8} S_g u = \frac{1}{4} K(\Lambda_\alpha, \tau_\alpha) u^5 + \frac{|\sigma_\alpha + DW_\alpha|^2}{8u^7} + \frac{|E_\alpha|^2 + |B_\alpha|^2}{8u_\alpha^3} + k_\alpha , \tag{0.7}$$

where $W_\alpha = W(E_\alpha, B_\alpha)$, $(\Lambda_\alpha)_\alpha$ and $(\tau_\alpha)_\alpha$ are sequences of real numbers, $(\sigma_\alpha)_\alpha$ is a sequence of 2-tensors, $(E_\alpha, B_\alpha)_\alpha$ is a sequence of electromagnetic fields, $(k_\alpha)_\alpha$ is a sequence of functions, and

$$\begin{aligned} &\Lambda_\alpha \to \Lambda \text{ and } \tau_\alpha \to \tau \text{ in } \mathbb{R} , \\ &\sigma_\alpha \to \sigma , (E_\alpha, B_\alpha) \to (E, B) , \text{ and } k_\alpha \to 0 \text{ in } C^0 \end{aligned} \tag{0.8}$$

as $\alpha \to +\infty$. Then, at least formally, (0.7) $\to$ (0.1) as $\alpha \to +\infty$. Following the terminology in Druet and Hebey [8] we say that

(i) (0.1) is *bounded and stable* if for any perturbation (0.7) of (0.1), and any sequence $(u_\alpha)_\alpha$ of solutions of (0.7), there exists a smooth positive solution of (0.1) such that, up to a subsequence, $u_\alpha \to u$ in $C^{1,\eta}$ for all $\eta \in (0,1)$,

(ii) (0.1) is *stable* if for any perturbation (0.7) of (0.1), and any bounded sequence $(u_\alpha)_\alpha$ in H^1 of solutions of (0.7), there exists a smooth positive solution of (0.1) such that, up to a subsequence, $u_\alpha \to u$ in $C^{1,\eta}$ for all $\eta \in (0,1)$,

Following standard terminology, (0.1) is said to be *compact* if for any sequence $(u_\alpha)_\alpha$ of solutions of (0.1), there holds that, up to a subsequence, $u_\alpha \to u$ in C^k for all k and some solution u of (0.1). As one can check from elliptic theory, the $C^{1,\eta}$-convergences in (i) and (ii) improve to $C^{k,\eta}$-convergences when the convergences in (0.8) improve accordingly. There holds that bounded stability implies stability, and since (0.1) can clearly be seen as a perturbation of itself, bounded stability implies compactness as well. On the contrary, an equation can be compact and unstable (see Druet and Hebey [9]). We adopt here the convention that an equation with no solution is stable (resp. bounded and stable) if it does not possess perturbations with H^1-bounded (resp. arbitrary) sequences of solutions. The theorem we prove concerning the stability of (0.1) is the following one.

Theorem 0.2. *Let (M,g) be a smooth closed 3-manifold, Λ, τ be constants, σ be a smooth $(2,0)$-tensor field, $(E,B) \not\equiv (0,0)$ be a nontrivial electromagnetic field, and $W = W(E,B)$ satisfy (0.6). The Lichnerowicz equation (0.1) is bounded and stable if $K \neq 0$, and stable and compact if $K = 0$, where $K = K(\Lambda,\tau)$ is given by (0.3).*

When $K = 0$, the equation cannot be bounded and stable without further assumptions as we can check from the toy model $S_g \equiv 1$, $\sigma + DW \equiv 0$, $|E|^2 + |B|^2 = 1$. Then

$$\Delta_g u_\alpha + \frac{1}{8} u_\alpha = \frac{1}{4} K_\alpha u_\alpha^5 + \frac{1}{8 u_\alpha^3}$$

for $u_\alpha \equiv \alpha$, where $K_\alpha = \frac{1}{2\alpha^5}\left(\alpha - \frac{1}{\alpha^3}\right)$. In particular, $K_\alpha \to 0$ as $\alpha \to +\infty$ while, obviously, in this example, $\|u_\alpha\|_{L^p} \to +\infty$ as $\alpha \to +\infty$ for any $1 \le p \le +\infty$. This contradicts bounded stability.

Some consequences of Theorem 0.2 are as follows. As a first consequence we assume (0.6). Then it easily follows from Theorem 0.2 that the solution in point (3) of Theorem 0.1 exists as well when $\theta = \theta_\star$. In other words, assuming (0.6), $Y(M,g) > 0$, and $K(\Lambda,\tau) > 0$, we get that there exists $\theta_\star > 0$ such that (0.5) has a solution when $\theta \le \theta_\star$, and no solution when $\theta > \theta_\star$.

As another consequence, if we still assume (0.6), that $Y(M,g) > 0$, and that $K(\Lambda,\tau) > 0$, then we get from Theorem 0.2 that we can actually compute the degree of our equation thanks to the existence and non existence parts in Theorem 0.1. More precisely we know from Theorem 0.1 that (0.5) has a solution when $\theta > 0$ is small, and no solution when θ is large, while we have by Theorem 0.2 that for any $\theta_1 < \theta_2$, the family of equations (0.5) inherits compactness for $\theta \in [\theta_1, \theta_2]$. Given $A > 0$, and $\eta \in (0,1)$, we define

$$\Omega_A = \left\{ u \in C^{2,\eta} \text{ s.t. } \|u\|_{C^{2,\eta}} < A \text{ and } \min_M u > \frac{1}{A} \right\} .$$

We fix $\theta_1 < 1 < \theta_2$ such that (0.5) has a solution when $\theta = \theta_1$ and no solution when $\theta = \theta_2$, and define $F_\theta : \overline{\Omega_A} \to C^{2,\eta}$ by

$$F_\theta(u) = u - \mathcal{L}^{-1}\left(\frac{1}{4}K(\Lambda,\tau)\,u^5 + \frac{|\sigma_\theta + DW_\theta|^2}{8u^7} + \frac{|E_\theta|^2 + |B_\theta|^2}{8u^3}\right),$$

where $\mathcal{L}(u) = \Delta_g u + \frac{1}{8}S_g u$. By the compactness of the family of equations (0.5) when $\theta \in [\theta_1, \theta_2]$, we get that there exists $A_0 \gg 1$ such that $F_\theta^{-1}(0) \subset \Omega_A$ for all $\theta \in [\theta_1, \theta_2]$ and all $A \geq A_0$. Then we can define the Leray-Schauder degree $\deg(F_\theta, \Omega_A, 0)$, and by homothopy invariance, since (0.5) has no solution when $\theta = \theta_2$, we get that

$$\deg(F_\theta, \Omega_A, 0) = 0 \tag{0.9}$$

for all $A \gg 1$. In particular, generically speaking, it follows from (0.9) that solutions of (0.5) come by pairs. A similar remark was carried out by Schoen [18] for the Yamabe equation whose degree, as computed in [18], turns out to be -1.

It can be noted that (0.1) arises in different Einstein-matter field theories like in the Einstein perfect fluids theory. We refer to Isenberg, Maxwell and Pollack [13] for more details on the building of these equations. Multiplicity for (0.1) when $(E,B) \equiv (0,0)$ is adressed in Premoselli [16].

1. Proof of Theorem 0.1

By conformal invariance, letting $\tilde{g} = \varphi^4 g$ for some smooth positive function φ, we get that u solves (0.5) if and only if $\frac{u}{\varphi}$ solves

$$\Delta_{\tilde{g}} u + \frac{1}{8}S_{\tilde{g}} u = \frac{1}{4}\left(\Lambda - \frac{1}{3}\tau^2\right)u^5 + \frac{|\sigma_\theta + DW_\theta|^2}{8\varphi^{12}u^7} + \frac{|E_\theta|^2 + |B_\theta|^2}{8\varphi^8 u^3}. \tag{1.1}$$

By the resolution of the Yamabe problem, see Aubin [1], Schoen [17] and Trüdinger [19], we can make $S_{\tilde{g}}$ constant. We separate the proof into the three cases of the theorem. The results in points (1) and (2) of Theorem 0.1 were expected from [6] and [12]. We provide here a slightly different, and probably easier, proof.

(1) Assume first that $Y(M,g) \leq 0$. We fix $\tilde{g}$ such that $S_{\tilde{g}} \leq 0$ is constant. Integrating (1.1) with respect to $dv_{\tilde{g}}$ it is clear that $K(\Lambda,\tau) < 0$ is a necessary condition for the existence of a solution to (1.1). Conversely, if we assume that $K(\Lambda,\tau) < 0$ and $Y(M,g) < 0$, then $u_- \equiv \varepsilon$ is a subsolution of (1.1) for $\varepsilon > 0$ sufficiently small, while $u_+ \equiv T$ is a supersolution of (1.1) for $T > 0$ sufficienlty large. By the sub and supersolution method we then get a solution $u_- \leq u \leq u_+$ of (1.1). In case $Y(M,g) = 0$, we let $s > 0$ and for $\delta > 0$ solve

$$\Delta_{\tilde{g}} u_\delta + s u_\delta = \frac{\delta}{4}K + \Phi,$$

where $K = K(\Lambda,\tau)$, and $\Phi = |\sigma_\theta + DW_\theta|^2 + |E_\theta|^2 + |B_\theta|^2$. We have that $u_\delta \to u_0$ in L^∞ as $\delta \to 0$, where u_0 solves $\Delta_{\tilde{g}} u_0 + s u_0 = \Phi$. Since $\Phi \geq 0$ and $\Phi \not\equiv 0$, we get by the maximum principle that $u_0 > 0$ in M. Hence $u_\delta > 0$ for $\delta > 0$ small, and for $t > 0$ sufficiently small it is easily checked that $u_- = t u_\delta > 0$ is a subsolution of (1.1). As before, by the sub and supersolution method, we obtain a solution $u_- \leq u \leq u_+$ of (1.1). This proves (1) in Theorem 0.1.

(2) Now we assume that $Y(M,g) > 0$ and that $K(\Lambda,\tau) \leq 0$. We fix $\tilde{g} \in [g]$ such that $S_{\tilde{g}} > 0$ is constant. Then, given $T > 0$ sufficiently large, $u_+ \equiv T$ is a supersolution of (1.1). For $\delta > 0$ we let u_δ solve

$$\Delta_{\tilde{g}} u_\delta + \frac{1}{8} S_{\tilde{g}} u_\delta = \frac{\delta}{4}\left(\Lambda - \frac{1}{3}\tau^2\right) + \frac{|E_\theta|^2 + |B_\theta|^2}{8\varphi^8} .$$

Clearly, $u_\delta \to u_0$ in L^∞ as $\delta \to 0$, where u_0 solves

$$\Delta_{\tilde{g}} u_0 + \frac{1}{8} S_{\tilde{g}} u_0 = \frac{|E_\theta|^2 + |B_\theta|^2}{8\varphi^8} . \tag{1.2}$$

There holds that $u_0 > 0$ in M since $(E,B) \not\equiv (0,0)$. In particular, $u_\delta > 0$ for $\delta > 0$ sufficiently small. We fix such a $\delta > 0$ sufficiently small, and let $u_\varepsilon = \varepsilon u_\delta$. Then

$$\begin{aligned} \Delta_{\tilde{g}} u_\varepsilon + \frac{1}{8} S_{\tilde{g}} u_\varepsilon &= \varepsilon \frac{\delta}{4}\left(\Lambda - \frac{1}{3}\tau^2\right) + \varepsilon \frac{|E_\theta|^2 + |B_\theta|^2}{8\varphi^8} \\ &\leq \frac{1}{4}\left(\Lambda - \frac{1}{3}\tau^2\right) u_\varepsilon^5 + \frac{|\sigma_\theta + DW_\theta|^2}{8\varphi^{12} u_\varepsilon^7} + \frac{|E_\theta|^2 + |B_\theta|^2}{8\varphi^8 u_\varepsilon^3} \end{aligned}$$

for $\varepsilon > 0$ sufficiently small since, for such ε's, $u_\varepsilon^5 \leq \varepsilon\delta$ and $\frac{1}{u_\varepsilon^3} \geq \varepsilon$ at any point in M. In particular, $u_- \equiv u_\varepsilon > 0$ is a subsolution of (1.1) when $\varepsilon > 0$ is sufficiently small. Then we can choose $\varepsilon > 0$ such that $u_- \leq u_+$ and by the sub and supersolution method we get a solution $u_- \leq u \leq u_+$ of (1.1). This proves (2) in Theorem 0.1.

(3) At last we assume that $Y(M,g) > 0$ and $K(\Lambda,\tau) > 0$. The existence of $\theta_\star$ can be obtained from the results in Hebey, Pacard and Pollack [11]. The proof in [11] relied on the mountain pass theorem. We provide here a much simpler argument, the price to pay being that we loose explicit control on $\theta_\star$ (a lower bound can be obtained from [11]). Here again we fix $\tilde{g} \in [g]$ such that $S_{\tilde{g}} > 0$ is constant. We let $u_0 > 0$ be given by (1.2) and define $u_\varepsilon = \varepsilon u_0$ for $\varepsilon > 0$. As above, u_ε turns out to be a subsolution of (1.1) for $\varepsilon > 0$ sufficiently small. Assume we have a solution u_θ of (1.1) for some $\theta > 0$. Then u_θ is a supersolution of (1.1) for $\theta' \leq \theta$. Choosing $\varepsilon > 0$ sufficiently small such that $u_\varepsilon \leq u_\theta$ in M, we obtain a solution of (1.1) for all $\theta' \leq \theta$ by the the sub and supersolution method. In particular, the set $\mathcal{S}$ consisting of the positive θ's for which (1.1) has a solution is either the empty set, or an interval like $(0,\theta_\star)$, where $\theta_\star \leq +\infty$. Let $\varepsilon_0 > 0$ be such that

$$\frac{1}{8} S_{\tilde{g}} \varepsilon_0 \geq \frac{1}{4} K(\Lambda,\tau)\varepsilon_0^5 + \varepsilon_0^2 .$$

By the definition of $E_\theta, B_\theta, W_\theta, \sigma_\theta$, we get that $u_+ \equiv \varepsilon_0$ is a supersolution of (1.1) for $\theta > 0$ sufficiently small. Choosing $\varepsilon > 0$ such that $u_\varepsilon \leq u_+$ we obtain a solution of (1.1) by the sub and supersolution method. In particular, there exists $\theta_0 > 0$ such that $(0,\theta_0) \subset \mathcal{S}$, and $\mathcal{S}$ is not empty. At this point, it remains to prove that $\theta_\star < +\infty$, and thus that there exists $\theta > 0$ for which (1.1) does not have a solution. We follow the argument in Hebey, Pacard and Pollack [11]. Integrating (1.1) there holds that

$$\begin{aligned} \frac{S_{\tilde{g}}}{8} \int_M u dv_{\tilde{g}} &= \frac{1}{4} K(\Lambda,\tau) \int_M u^5 dv_{\tilde{g}} + \theta^2 \int_M \frac{\Phi_1}{u^7} dv_{\tilde{g}} + \theta^2 \int_M \frac{\Phi_2}{u^3} dv_{\tilde{g}} \\ &\geq \frac{1}{4} K(\Lambda,\tau) \int_M u^5 dv_{\tilde{g}} + \theta^2 \int_M \frac{\Phi_2}{u^3} dv_{\tilde{g}} , \end{aligned} \tag{1.3}$$

where $\Phi_1, \Phi_2 \geq 0$ are nonnegative functions and $\Phi_2 \not\equiv 0$. By Hölder's inequality, $\int u \leq V_{\tilde{g}}^{4/5} \left(\int u^5\right)^{1/5}$, where $V_{\tilde{g}}$ is the volume of M with respect to $\tilde{g}$, while

$$\begin{aligned}\int_M \Phi_2^{5/12} dv_{\tilde{g}} &= \int_M \Phi_2^{5/12} \frac{1}{u^{5/4}} u^{5/4} dv_{\tilde{g}} \\ &\leq \left(\int_M \frac{\Phi_2}{u^3} dv_{\tilde{g}}\right)^{5/12} \left(\int_M u^{15/7} dv_{\tilde{g}}\right)^{7/12} \\ &\leq V_{\tilde{g}}^{1/3} \left(\int_M \frac{\Phi_2}{u^3} dv_{\tilde{g}}\right)^{5/12} \left(\int_M u^5 dv_{\tilde{g}}\right)^{1/4}\end{aligned} \tag{1.4}$$

Letting $X = \left(\int_M u^5 dv_{\tilde{g}}\right)^{4/5}$, it follows from (1.3) and (1.4) that

$$\begin{aligned}\frac{\sqrt{K(\Lambda,\tau)} \left(\int_M \Phi_2^{5/12} dv_{\tilde{g}}\right)^{6/5} \theta}{V_{\tilde{g}}^{2/5}} &\leq \frac{K(\Lambda,\tau)}{4} X + \frac{\theta^2 \left(\int_M \Phi_2^{5/12} dv_{\tilde{g}}\right)^{12/5}}{V_{\tilde{g}}^{4/5} X} \\ &\leq \frac{S_{\tilde{g}}}{8} V_{\tilde{g}}^{4/5} .\end{aligned} \tag{1.5}$$

As is easily checked, (1.5) is impossible when $\theta > 0$ is large. In particular, (1.1) does not have a solution when $\theta > 0$ is large and we get that $\theta_\star < +\infty$. This proves (3) in Theorem 0.1.

2. Proof of Theorem 0.2 when $K < 0$

We assume that $K(\Lambda, \tau) < 0$ and let $(u_\alpha)_\alpha$ be a sequence of solutions of (0.7). Then

$$\Delta_g u_\alpha + \frac{1}{8} S_g u_\alpha = \frac{1}{4} K_\alpha u_\alpha^5 + \frac{\Phi_\alpha}{u_\alpha^7} + \frac{\Psi_\alpha}{u_\alpha^3} + k_\alpha , \tag{2.1}$$

where $K_\alpha = \Lambda_\alpha - \frac{1}{3}\tau_\alpha^2$ is such that $K_\alpha \to K(\Lambda,\tau)$ as $\alpha \to +\infty$, $\Phi_\alpha, \Psi_\alpha \geq 0$, and $\Phi_\alpha \to \Phi$, $\Psi_\alpha \to \Psi$ and $k_\alpha \to 0$ in C^0 as $\alpha \to +\infty$ for some smooth functions Φ, Ψ with $\Psi \not\equiv 0$. In the present setting,

$$\Phi_\alpha = \frac{|\sigma_\alpha + DW_\alpha|^2}{8} \text{ and } \Psi_\alpha = \frac{|E_\alpha|^2 + |B_\alpha|^2}{8} . \tag{2.2}$$

Let $x_\alpha \in M$ be such that $u_\alpha(x_\alpha) = \max_M u_\alpha$. Since $K(\Lambda,\tau) < 0$, there holds that $K_\alpha < 0$ for $\alpha \gg 1$. Then,

$$\frac{1}{8} S_g u_\alpha(x_\alpha) + \frac{1}{4}|K_\alpha| u_\alpha(x_\alpha)^5 \leq \frac{\Phi_\alpha(x_\alpha)}{u_\alpha(x_\alpha)^7} + \frac{\Psi_\alpha(x_\alpha)}{u_\alpha(x_\alpha)^3} + \|k_\alpha\|_{L^\infty}$$

and we get that there exists $C > 0$ such that $u_\alpha \leq C$ in M for all α. Given $s > 0$ we let $h_s = \frac{1}{8} S_g + s$, and we fix $s \gg 1$ such that $h_s \geq 1$. Given $\delta > 0$ we solve

$$\begin{cases} \Delta_g u_{\alpha,\delta} + h_s u_{\alpha,\delta} = \Phi_\alpha + \Psi_\alpha + \frac{\delta}{4} K_\alpha \\ \Delta_g u_\delta + h_s u_\delta = \Phi + \Psi + \frac{\delta}{4} K \\ \Delta_g r_\alpha + h_s r_\alpha = k_\alpha , \end{cases} \tag{2.3}$$

where $K = K(\Lambda,\tau)$. Then $u_{\alpha,\delta} \to u_\delta$ in L^∞ as $\alpha \to +\infty$, while $u_\delta \to u_0$ in L^∞ as $\delta \to 0$, where

$$\Delta_g u_0 + h_s u_0 = \Phi + \Psi .$$

Since $\Phi, \Psi \geq 0$ and $\Psi \not\equiv 0$, there holds that $u_0 > 0$ in M. In particular, $u_\delta > 0$ in M for $\delta > 0$ sufficiently small, and fixing $\delta > 0$ small, we get that there exists

$\varepsilon_0 > 0$ such that $u_{\alpha,\delta} \geq \varepsilon_0$ for all $\alpha \gg 1$. Now we define $\varphi_\alpha = tu_{\alpha,\delta} + r_\alpha$ for $t > 0$, where r_α is as in (2.3). There holds that $r_\alpha \to 0$ in L^∞ as $\alpha \to +\infty$. Given $t > 0$ sufficiently small we have that

$$\begin{aligned}\Delta_g\varphi_\alpha + h_s\varphi_\alpha &= t\Phi_\alpha + t\Psi_\alpha + \frac{\delta}{4}tK_\alpha + k_\alpha\\ &\leq \frac{\Phi_\alpha}{\varphi_\alpha^7} + \frac{\Psi_\alpha}{\varphi_\alpha^3} + \frac{1}{4}K_\alpha\varphi_\alpha^5 + k_\alpha\end{aligned}$$

for all $\alpha \gg 1$, since $K_\alpha < 0$ for $\alpha \gg 1$, $u_{\alpha,\delta} \to u_\delta$ in L^∞ as $\alpha \to +\infty$, and $u_{\alpha,\delta} \geq \varepsilon_0 > 0$ and $r_\alpha \to 0$ in L^∞. We fix $t > 0$ sufficiently small. Then, for $\alpha \gg 1$,

$$\begin{aligned}&\Delta_g\left(u_\alpha - \varphi_\alpha\right) + h_s\left(u_\alpha - \varphi_\alpha\right)\\ &\geq \frac{1}{4}K_\alpha\left(u_\alpha^5 - \varphi_\alpha^5\right) + \Phi_\alpha\left(u_\alpha^{-7} - \varphi_\alpha^{-7}\right) + \Psi_\alpha\left(u_\alpha^{-3} - \varphi_\alpha^{-3}\right)\\ &\geq 0\end{aligned}$$

at any point such that $u_\alpha \leq \varphi_\alpha$. Hence, $\varphi_\alpha \leq u_\alpha$ in M for all $\alpha \gg 1$ by the maximum principle. Since $\varphi_\alpha \geq \frac{1}{2}t\varepsilon_0$ for $\alpha \gg 1$, we get that there exists $C > 1$ such that

$$\frac{1}{C} \leq u_\alpha \leq C \tag{2.4}$$

in M for all α. By (2.1) and elliptic theory we then get that the u_α's are bounded in $C^{1,\eta}$ for $0 < \eta < 1$. In particular, up to a subsequence, $u_\alpha \to u$ in $C^{1,\eta'}$ for $\eta' \in (0,\eta)$ and some smooth positive solution u of (0.1). This proves the stability of (0.1).

3. Proof of Theorem 0.2 when $K = 0$

We prove first stability. We assume that $K(\Lambda,\tau) = 0$ and let $(u_\alpha)_\alpha$ be a bounded sequence in H^1 of solutions of (0.7). Then

$$\Delta_g u_\alpha + \frac{1}{8}S_g u_\alpha = \frac{1}{4}K_\alpha u_\alpha^5 + \frac{\Phi_\alpha}{u_\alpha^7} + \frac{\Psi_\alpha}{u_\alpha^3} + k_\alpha\ , \tag{3.1}$$

where $K_\alpha = \Lambda_\alpha - \frac{1}{3}\tau_\alpha^2$ is such that $K_\alpha \to 0$ as $\alpha \to +\infty$, $\Phi_\alpha, \Psi_\alpha \geq 0$, and $\Phi_\alpha \to \Phi$, $\Psi_\alpha \to \Psi$ and $k_\alpha \to 0$ in C^0 as $\alpha \to +\infty$ for some smooth functions Φ, Ψ with $\Psi \not\equiv 0$. Namely, Φ_α and Ψ_α are given by (2.2). Moreover we have here that $\|u_\alpha\|_{H^1} \leq C$ for some $C > 0$ and all α. Replacing K_α by $\tilde{K}_\alpha = \min(0, K_\alpha)$ in (2.3), we get as above that there exists $\varepsilon_0 > 0$ such that $u_\alpha \geq \varepsilon_0$ in M for all α. Now we let $x_\alpha \in M$ be such that $u_\alpha(x_\alpha) = \max_M u_\alpha$. By contradiction we assume that $u_\alpha(x_\alpha) \to +\infty$ as $\alpha \to +\infty$. Let $\mu_\alpha > 0$ be given by $\mu_\alpha^{-1/2} = u_\alpha(x_\alpha)$, and let $\tilde{u}_\alpha$ be given by

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{1}{2}} u_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$$

for $x \in \mathbb{R}^3$. Let also $\tilde{g}_\alpha = \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x)$. There holds that

$$\Delta_{\tilde{g}_\alpha}\tilde{u}_\alpha + \mu_\alpha^2 h_\alpha \tilde{u}_\alpha = K_\alpha \tilde{u}_\alpha^5 + \mu_\alpha^{5/2}\tilde{R}_\alpha\ , \tag{3.2}$$

where

$$h_\alpha(x) = \frac{1}{8}S_g\left(\exp_{x_\alpha}(\mu_\alpha x)\right)\ ,\ \tilde{R}_\alpha(x) = R_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)\ ,$$

and R_α is some function such that $\|R_\alpha\|_{L^\infty} \leq C$ for some $C > 0$ and all α. By construction, $0 \leq \tilde{u}_\alpha \leq 1$. By elliptic theory and (3.2) we then get that there exists $\tilde{u} : \mathbb{R}^3 \to \mathbb{R}$, $\tilde{u} \in \dot{H}^1$, such that $\tilde{u}_\alpha \to \tilde{u}$ in $C^2_{loc}(\mathbb{R}^3)$. Then, $0 \leq \tilde{u} \leq 1$, $\tilde{u}(0) = 1$, and by (3.2), since $K_\alpha \to 0$ and $\mu_\alpha \to 0$, we get that $\tilde{u}$ is harmonic. A contradiction with Liouville's theorem. This implies that there exists $C > 0$ such that $u_\alpha \leq C$ for all α. We conclude to the stability of (0.1) as in the preceeding section.

Concerning compactness, we let $(u_\alpha)_\alpha$ be any sequence of solutions of (0.1). By Theorem 0.1 this implies that $Y(M,g) > 0$ (since if not the case, the equation does not have any solution). As above when discussing stability we obtain that there exists $\varepsilon_0 > 0$ such that $u_\alpha \geq \varepsilon_0$ in M for all α. The proof can even be made much simpler. We just need to solve

$$\Delta_g u_0 + h_s u_0 = \Phi + \Psi \, .$$

Then we can note that $u_0 > 0$, that

$$\Delta_g \varphi + h_s \varphi \leq \frac{\Phi}{\varphi^7} + \frac{\Psi}{\varphi^3}$$

for $\varphi = t u_\delta$ when $0 < t \ll 1$, and that

$$\Delta_g (u_\alpha - \varphi) + h_s (u_\alpha - \varphi) \geq 0$$

at any point where $u_\alpha \leq \varphi$. We conclude as in the preceeding section to the existence of ε_0. Now we let $\tilde{g} \in [g]$ be such that $S_{\tilde{g}} > 0$ is constant, and let $\tilde{u}_\alpha = \frac{u_\alpha}{\varphi}$. Then $\tilde{u}_\alpha$ solves

$$\Delta_{\tilde{g}} \tilde{u}_\alpha + \frac{1}{8} S_{\tilde{g}} \tilde{u}_\alpha = \frac{|\sigma + DW|^2}{8\varphi^{12}\tilde{u}_\alpha^7} + \frac{|E|^2 + |B|^2}{8\varphi^8 \tilde{u}_\alpha^3} \, . \tag{3.3}$$

If x_α is a point where $\tilde{u}_\alpha$ is maximum, since $\Delta_{\tilde{g}} \tilde{u}_\alpha(x_\alpha) \geq 0$ and $S_{\tilde{g}} > 0$, we get that $\tilde{u}_\alpha(x_\alpha) \leq C$ for some $C > 0$ and all α. In particular, there exists $C > 0$ such that $u_\alpha \leq C$ for all α. This, together with the lower bound $u_\alpha \geq \varepsilon_0$, implies the compactness of (0.1) thanks to standard elliptic theory.

As a final remark on the $K = 0$ case, suppose $Y(M,g) \leq 0$. By Theorem 0.1, there exists $u_\alpha > 0$ which solves (0.1) for any $K_\alpha < 0$. Let $(K_\alpha)_\alpha$ be such that $K_\alpha < 0$ for all α and $K_\alpha \to 0$ as $\alpha \to +\infty$. Choosing $\tilde{g} \in [g]$ such that $S_{\tilde{g}} \leq 0$, and up to changing u_α into $\tilde{u}_\alpha = \varphi^{-1} u_\alpha$ for some smooth positive function φ, there holds that

$$\Delta_{\tilde{g}} \tilde{u}_\alpha + \frac{1}{8} S_{\tilde{g}} \tilde{u}_\alpha = \frac{1}{4} K_\alpha \tilde{u}_\alpha^5 + \frac{|\sigma + DW|^2}{8\varphi \tilde{u}_\alpha^7} + \frac{|E|^2 + |B|^2}{8\varphi \tilde{u}_\alpha^3} \tag{3.4}$$

for all α. Integrating (3.4) we see that, necessarily, $\int_M \tilde{u}_\alpha^5 dv_g \to +\infty$ as $\alpha \to +\infty$. In particular, (0.1) when $K = 0$ possesses perturbations which themselves possess sequences of solutions with unbounded energy. This provides another illustration, in addition to the toy model mentioned in the introduction, of the fact that (0.1) when $K = 0$ cannot be bounded and stable.

4. Proof of Theorem 0.2 when $K > 0$

We assume here that $K > 0$ and follow the more involved analysis in Druet and Hebey [8]. We let $(u_\alpha)_\alpha$ be a sequence of solutions of (0.7). Then

$$\Delta_g u_\alpha + \frac{1}{8} S_g u_\alpha = \frac{1}{4} K_\alpha u_\alpha^5 + \frac{\Phi_\alpha}{u_\alpha^7} + \frac{\Psi_\alpha}{u_\alpha^3} + k_\alpha \, , \tag{4.1}$$

where $K_\alpha = \Lambda_\alpha - \frac{1}{3}\tau_\alpha^2$ is such that $K_\alpha \to K$ as $\alpha \to +\infty$, $K > 0$, $\Phi_\alpha, \Psi_\alpha \geq 0$, and $\Phi_\alpha \to \Phi$, $\Psi_\alpha \to \Psi$ and $k_\alpha \to 0$ in C^0 as $\alpha \to +\infty$ for some smooth functions Φ, Ψ

with $\Psi \not\equiv 0$. Namely, Φ_α and Ψ_α are given by (2.2). We separate the proof into several lemmas. Following the argument in Section 2 we first prove that the u_α's are bounded from below far from 0.

Lemma 4.1. *Let $(u_\alpha)_\alpha$ solves (4.1). There exists $\varepsilon_0 > 0$ such that $u_\alpha \geq \varepsilon_0$ for all α.*

Proof of Lemma 4.1. Given $s > 0$ we let $h_s = \frac{1}{8}S_g + s$, and we fix $\theta \gg 1$ such that $h_s \geq 1$. Given $\delta > 0$ we solve

$$\begin{cases} \Delta_g u_{\alpha,\delta} + h_s u_{\alpha,\delta} = \Phi_\alpha + \Psi_\alpha \\ \Delta_g u_\delta + h_s u_\delta = \Phi + \Psi \\ \Delta_g r_\alpha + h_s r_\alpha = k_\alpha \ . \end{cases} \tag{4.2}$$

Then $u_{\alpha,\delta} \to u_\delta$ in L^∞ as $\alpha \to +\infty$, while $u_\delta \to u_0$ in L^∞ as $\delta \to 0$, where

$$\Delta_g u_0 + h_s u_0 = \Phi + \Psi \ .$$

Since $\Phi, \Psi \geq 0$ and $\Psi \not\equiv 0$, there holds that $u_0 > 0$ in M. In particular, $u_\delta > 0$ in M for $\delta > 0$ sufficiently small, and fixing $\delta > 0$ small, we get that there exists $\varepsilon_0 > 0$ such that $u_{\alpha,\delta} \geq \varepsilon_0$ for all $\alpha \gg 1$. Now we define

$$\varphi_\alpha = t u_{\alpha,\delta} + r_\alpha$$

for $t > 0$, where r_α is as in (4.2). There holds that $r_\alpha \to 0$ in L^∞ as $\alpha \to +\infty$. Given $t > 0$ sufficiently small we have that

$$\Delta_g \varphi_\alpha + h_s \varphi_\alpha \leq \frac{\Phi_\alpha}{\varphi_\alpha^7} + \frac{\Psi_\alpha}{\varphi_\alpha^3} + k_\alpha$$

for all $\alpha \gg 1$. We fix $t > 0$ sufficiently small. Then, for $\alpha \gg 1$,

$$\begin{aligned} &\Delta_g \left(u_\alpha - \varphi_\alpha\right) + h_s \left(u_\alpha - \varphi_\alpha\right) \\ &\geq \frac{1}{4} K_\alpha u_\alpha^5 + \Phi_\alpha \left(u_\alpha^{-7} - \varphi_\alpha^{-7}\right) + \Psi_\alpha \left(u_\alpha^{-3} - \varphi_\alpha^{-3}\right) \\ &\geq 0 \end{aligned}$$

at any point such that $u_\alpha \leq \varphi_\alpha$. Hence, $\varphi_\alpha \leq u_\alpha$ in M for all $\alpha \gg 1$ by the maximum principle. Since $\varphi_\alpha \geq \frac{1}{2} t\varepsilon_0$ for $\alpha \gg 1$, we get a lower bound for the u_α's and this ends the proof of the lemma. ☐

Thanks to Lemma 4.1, there holds that $|\Delta_g u_\alpha| \leq C u_\alpha^5$ in M for all α. We assume in what follows that the u_α's blow up. By elliptic theory, the sole C^0-norm of the u_α's is involved. In other words, we assume that

$$\lim_{\alpha \to +\infty} \|u_\alpha\|_{L^\infty} = +\infty \ . \tag{4.3}$$

Then, as is by now classical in blow-up theory, there exists $C > 0$ such that for any α, there exist $N_\alpha \in \mathbb{N}^\star$ and a set $\mathcal{S}_\alpha$ of N_α critical points of u_α, denoted by

$$\mathcal{S}_\alpha = \left\{(x_{1,\alpha}, x_{2,\alpha}, \ldots, x_{N_\alpha,\alpha})\right\} \ ,$$

such that

$$d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)^{\frac{1}{2}} u_\alpha\left(x_{i,\alpha}\right) \geq 1 \tag{4.4}$$

for all i, $j \in \{1, \ldots, N_\alpha\}$, $i \neq j$, and

$$\left(\min_{i=1,\ldots,N_\alpha} d_g\left(x_{i,\alpha}, x\right)\right)^{\frac{1}{2}} u_\alpha(x) \leq C \tag{4.5}$$

for all $x \in M$ and all α. Now we let $(x_\alpha)_\alpha$ be a sequence of points in M and $(\rho_\alpha)_\alpha$ be a sequence of positive real numbers with $0 < \rho_\alpha < \frac{1}{7} i_g(M)$ such that

$$\begin{aligned} & x_\alpha \text{ is a critical point of } u_\alpha \text{ , and} \\ & d_g(x_\alpha, x)^{\frac{1}{2}} u_\alpha(x) \le C \text{ for all } x \in B_{x_\alpha}(7\rho_\alpha) \end{aligned} \tag{4.6}$$

for all α. By (4.4) and (4.5), we get that (4.6) holds true for $x_\alpha = x_{i_\alpha,\alpha}$ where $(i_\alpha)_\alpha$ is a sequence of integers in $\{1,\dots,N_\alpha\}$ up to take ρ_α sufficiently small, and in particular less than $1/8^{th}$ of the minimum distance between the $x_{i,\alpha}$'s and x_α for $i \neq i_\alpha$. In addition to (4.6) we consider the assumption that

$$\rho_\alpha^{\frac{1}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha \to +\infty \tag{4.7}$$

as $\alpha \to +\infty$. Rescaling arguments, as in Claim 1 of Druet [7], then give that, up to a subsequence, $x_\alpha \to x_0$ as $\alpha \to +\infty$ for some $x_0 \in M$, and

$$\mu_\alpha^{\frac{1}{2}} u_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right) \to \overline{u}(x) \tag{4.8}$$

in $C^1_{loc}(\mathbb{R}^3)$ as $\alpha \to +\infty$, where $\mu_\alpha > 0$ is given by $u_\alpha(x_\alpha) = \mu_\alpha^{-1/2}$, and

$$\overline{u}(x) = \left(\frac{1}{1+\frac{K|x|^2}{12}}\right)^{1/2} .$$

In particular, there holds that $\frac{\rho_\alpha}{\mu_\alpha} \to +\infty$ as $\alpha \to +\infty$, and $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Now we define $\varphi_\alpha : (0,\rho_\alpha) \mapsto \mathbb{R}^+$ by

$$\varphi_\alpha(r) = \frac{1}{|\partial B_{x_\alpha}(r)|_g} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g , \tag{4.9}$$

where $|\partial B_{x_\alpha}(r)|_g$ is the volume of $\partial B_{x_\alpha}(r)$ for the metric induced by g. By (4.8),

$$(\mu_\alpha r)^{\frac{1}{2}} \varphi_\alpha(\mu_\alpha r) \to r^{\frac{1}{2}} \left(1 + \frac{Kr^2}{12}\right)^{-\frac{1}{2}} \tag{4.10}$$

in $C^1_{loc}\big([0,+\infty)\big)$ as $\alpha \to +\infty$. We define $r_\alpha \in (2R_0\mu_\alpha, \rho_\alpha]$ by

$$\begin{aligned} r_\alpha = \sup\Big\{ & r \in (2R_0\mu_\alpha,\rho_\alpha) \text{ s.t.} \\ & s^{\frac{1}{2}}\varphi_\alpha(s) \text{ is nonincreasing in } (2R_0\mu_\alpha, r)\Big\} , \end{aligned} \tag{4.11}$$

where $R_0^2 = \frac{12}{K}$ and φ_α is as in (4.9). We know thanks to (4.10) that

$$\frac{r_\alpha}{\mu_\alpha} \to +\infty \tag{4.12}$$

as $\alpha \to +\infty$. Also we have that

$$\begin{aligned} & r^{\frac{1}{2}}\varphi_\alpha \text{ is non-increasing in } (2R_0\mu_\alpha, r_\alpha) \text{ , and} \\ & \text{and that } \left(r^{\frac{1}{2}}\varphi_\alpha(r)\right)'(r_\alpha) = 0 \text{ if } r_\alpha < \rho_\alpha . \end{aligned} \tag{4.13}$$

Now we prove that the following sharp pointwise estimates on the u_α's hold true.

Lemma 4.2. *Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that (4.6) and (4.7) hold true. Up to passing to a subsequence, $\rho_\alpha \to 0$, $\rho_\alpha^{1/2} u_\alpha(x_\alpha) \to +\infty$ and*

$$u_\alpha(x_\alpha)\,\rho_\alpha u_\alpha\left(\exp_{x_\alpha}(\rho_\alpha x)\right) \to \frac{\lambda}{|x|} + H(x)$$

in $C^2_{loc}(B_0(5)\setminus\{0\})$ as $\alpha \to +\infty$ for some $\lambda > 0$ and some harmonic function H in $B_0(5)$ which satisfies $H(0) = 0$.

Proof of Lemma 4.2. Mimicking the analysis in Druet and Hebey [8] we get that there exists $C > 0$ such that

$$u_\alpha(x) + d_g(x_\alpha, x)\,|\nabla u_\alpha(x)| \le C\mu_\alpha^{\frac{1}{2}} d_g(x_\alpha, x)^{-1} \tag{4.14}$$

for all $x \in B_{x_\alpha}(6r_\alpha)\setminus\{x_\alpha\}$. As a consequence, $r_\alpha^2 = O(\mu_\alpha)$, where μ_α is as in (4.8). First we prove that, after passing to a subsequence,

$$\mu_\alpha^{-\frac{1}{2}} r_\alpha u_\alpha\left(\exp_{x_\alpha}(r_\alpha x)\right) \to \frac{R_0}{|x|} + H(x) \tag{4.15}$$

in $C^2_{loc}(B_0(5)\setminus\{0\})$ as $\alpha \to +\infty$ where H is some harmonic function in $B_0(5)$. Also we prove that if $r_\alpha < \rho_\alpha$, then we have that $H(0) = R_0$. We set, for $x \in \mathbb{R}^3$,

$$\hat{u}_\alpha(x) = \mu_\alpha^{-\frac{1}{2}} r_\alpha u_\alpha\left(\exp_{x_\alpha}(r_\alpha x)\right) \text{ and } \hat{g}_\alpha(x) = \left(\exp^\star_{x_\alpha} g\right)(r_\alpha x)\ .$$

We know that $\hat{g}_\alpha \to \xi$ in $C^2_{loc}\left(\mathbb{R}^3\right)$ since $r_\alpha \to 0$ as $\alpha \to +\infty$. We also know thanks to (4.14) that $\hat{u}_\alpha(x) \le C|x|^{-1}$ in $B_0(6)\setminus\{0\}$. Using (4.1) we then get that $\Delta_{\hat{g}_\alpha}\hat{u}_\alpha = \hat{F}_\alpha$ in $B_0(6)$, where

$$\left|\hat{F}_\alpha(x)\right| \le C\frac{\mu_\alpha^2}{r_\alpha^2}|x|^{-5}$$

in $B_0(6)\setminus\{0\}$ thanks to Lemma 4.1. By (4.12) and standard elliptic theory we easily get that $\hat{u}_\alpha \to \hat{U}$ in $C^1_{loc}(B_0(5)\setminus\{0\})$ as $\alpha \to +\infty$, where $\Delta_\xi \hat{U} = 0$ in $B_0(5)\setminus\{0\}$. Moreover, we have that $0 \le \hat{U}(x) \le C_4|x|^{-1}$. We can write that

$$\hat{U}(x) = \frac{\lambda}{|x|^{n-2}} + H(x)$$

for some $\lambda \ge 0$ and some function H harmonic in $B_0(5)$. In order to get (4.15) it remains to prove that $\lambda = R_0$. This can be done by integrating the equation $\Delta_{\hat{g}_\alpha}\hat{u}_\alpha = \hat{F}_\alpha$ satisfied by $\hat{u}_\alpha$ on $B_0(1)$. Indeed, integrating the equation we get that

$$-\int_{\partial B_0(1)} \partial_\nu \hat{u}_\alpha d\sigma_{\hat{g}_\alpha} = \int_{B_0(1)} \hat{F}_\alpha dv_{\hat{g}_\alpha}\ .$$

Thanks to (4.8) and (4.14) we easily obtain that

$$\int_{B_0(1)} \hat{F}_\alpha dv_{\hat{g}_\alpha} \to \frac{K}{4}\int_{\mathbb{R}^3}\left(1 + \frac{|x|^2}{R_0^2}\right)^{-\frac{5}{2}} dx$$

as $\alpha \to +\infty$. It is sufficient to remark that

$$\frac{K}{4}\int_{\mathbb{R}^3}\left(1 + \frac{|x|^2}{R_0^2}\right)^{-\frac{5}{2}} dx = \omega_2 R_0$$

and that

$$-\int_{\partial B_0(1)} \partial_\nu \hat{u}_\alpha\, d\sigma_{\hat{g}_\alpha} \to -\lambda\omega_2$$

as $\alpha \to +\infty$ to conclude that $\lambda = R_0$. In case $r_\alpha < \rho_\alpha$, we can use (4.13) to get that $\left(r^{1/2}\varphi(r)\right)'(1) = 0$, where

$$\varphi(r) = \frac{1}{\omega_2 r^2}\int_{\partial B_0(r)} \hat{U}\, d\sigma = \frac{R_0}{r} + H(0)\ .$$

We conclude that $H(0) = R_0$ if $r_\alpha < \rho_\alpha$. In particular, in order to end the proof of Lemma 4.2, it remains to prove that $H(0) = 0$. For this we apply the Pohozaev identity in Druet and Hebey [8]. Let X_α be the vector field which coordinates in the exponential chart at x_α are $X_\alpha^i = x^i$, applying the identity in $\Omega_\alpha = B_{x_\alpha}(r_\alpha)$, we get that

$$\begin{aligned}&\int_{\Omega_\alpha}\left(\nabla u_\alpha\left(X_\alpha\right) + \frac{1}{2}u_\alpha\right)\Delta_g u_\alpha dv_g\\ &= O\left(\int_{\Omega_\alpha} d_g\left(x_\alpha, x\right)^2\left|\nabla u_\alpha\right|^2 dv_g\right)\\ &+\int_{\partial\Omega_\alpha}\left(\frac{1}{2}\left(X_\alpha,\nu\right)_g\left|\nabla u_\alpha\right|^2 - \nabla u_\alpha\left(X_\alpha\right)\partial_\nu u_\alpha - \frac{1}{2}u_\alpha\partial_\nu u_\alpha\right)d\sigma_g\ ,\end{aligned} \tag{4.16}$$

where ν is the unit outer normal of $\partial\Omega_\alpha$. By (4.15),

$$\begin{aligned}&\int_{\partial\Omega_\alpha}\left(\frac{1}{2}\left(X_\alpha,\nu\right)_g\left|\nabla u_\alpha\right|^2 - \nabla u_\alpha\left(X_\alpha\right)\partial_\nu u_\alpha - \frac{1}{2}u_\alpha\partial_\nu u_\alpha\right)d\sigma_g\\ &= \left(\frac{1}{2}\omega_2 R_0 H(0) + o(1)\right)\mu_\alpha r_\alpha^{-1}\ .\end{aligned} \tag{4.17}$$

Since $r_\alpha^2 = O(\mu_\alpha)$ we can write that $\mu_\alpha r_\alpha = o(\mu_\alpha r_\alpha^{-1})$. Then, by Lemma 4.1, (4.8) and (4.14), and thanks to the equation satisfied by the u_α's, there holds that

$$\begin{aligned}&\int_{\Omega_\alpha} d_g\left(x_\alpha, x\right)^2\left|\nabla u_\alpha\right|^2 dv_g = o\left(\mu_\alpha r_\alpha^{-1}\right)\ ,\ \text{and}\\ &\int_{\Omega_\alpha}\left(\nabla u_\alpha\left(X_\alpha\right) + \frac{1}{2}u_\alpha\right)\Delta_g u_\alpha dv_g = o\left(\mu_\alpha r_\alpha^{-1}\right)\ .\end{aligned} \tag{4.18}$$

Plugging (4.17) and (4.18) into (4.16) we get that $H(0) = 0$. This ends the proof of Lemma 4.2. □

Now we return to the situation in the beginning of the section and let

$$d_\alpha = \min_{1\le i<j\le N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)\ , \tag{4.19}$$

where N_α and the $x_{i,\alpha}$'s are as in (4.4) and (4.5). By Lemma 4.2 we may assume $N_\alpha \ge 2$. Up to reorder the points, we may also assume that

$$d_\alpha = d_g\left(x_{1,\alpha}, x_{2,\alpha}\right) \le d_g\left(x_{1,\alpha}, x_{3,\alpha}\right) \le \cdots \le d_g\left(x_{1,\alpha}, x_{N_\alpha,\alpha}\right)\ . \tag{4.20}$$

Thanks to Lemma 4.2 we can now end the proof of Theorem 0.2.

Proof of Theorem 0.2. We prove both that $d_\alpha \not\to 0$ and that $d_\alpha \to 0$. First we claim that there necessarily holds that $d_\alpha \not\to 0$ and thus that, up to a subsequence, $d_\alpha \ge d$ for some $d > 0$. In order to prove the existence of d we proceed by contradiction and assume that $d_\alpha \to 0$ as $\alpha \to +\infty$. Let $\delta \in (0, \frac{1}{2}i_g)$ be given. For $x \in B_0(\delta d_\alpha^{-1})$, we let

$$\tilde{u}_\alpha(x) = d_\alpha^{1/2} u_\alpha\left(\exp_{x_{1,\alpha}}(d_\alpha x)\right)\ . \tag{4.21}$$

Let also $\tilde{g}_\alpha(x) = \left(\exp^\star_{x_{1,\alpha}} g\right)(d_\alpha x)$. Then

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + d_\alpha^2 h_\alpha \tilde{u}_\alpha = \frac{1}{4} K_\alpha \tilde{u}_\alpha^5 + d_\alpha^{5/2} \tilde{R}_\alpha \ , \tag{4.22}$$

where $h_\alpha(x) = \frac{1}{8} S_g\left(\exp_{x_\alpha}(d_\alpha x)\right)$, $\tilde{R}_\alpha(x) = R_\alpha\left(\exp_{x_\alpha}(d_\alpha x)\right)$, and $\|R_\alpha\|_{L^\infty} \le C$ for some $C > 0$. We clearly have that $\tilde{g}_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^3)$ as $\alpha \to +\infty$. Given $R > 0$ we let $1 \le N_{R,\alpha} \le N_\alpha$ be such that $d_g(x_{1,\alpha}, x_{i,\alpha}) \le R d_\alpha$ for all $1 \le i \le N_{R,\alpha}$, and $d_g(x_{1,\alpha}, x_{i,\alpha}) > R d_\alpha$ for all $N_{R,\alpha} + 1 \le i \le N_\alpha$. We have that $N_{R,\alpha} \ge 2$ for all $R > 1$, and $(N_{R,\alpha})_\alpha$ is uniformly bounded for all $R > 0$. Mimicking the arguments in Druet and Hebey [8], see also Druet, Hebey and Vétois [10], given $R > 0$, there holds that

$$\begin{aligned} &\text{either } \tilde{u}_\alpha(\tilde{x}_{i,\alpha}) = O(1) \text{ for all } 1 \le i \le N_{R,\alpha} \ , \\ &\text{or } \tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \to +\infty \text{ as } \alpha \to +\infty \text{ for all } 1 \le i \le N_{R,\alpha} \ , \end{aligned} \tag{4.23}$$

where the $\tilde{u}_\alpha$'s are as in (4.21), and

$$\tilde{x}_{i,\alpha} = \frac{1}{d_\alpha} \exp^{-1}_{x_{1,\alpha}}(x_{i,\alpha}) \ . \tag{4.24}$$

Now we split the proof into the study of two cases. In the first case we assume that there exist $R > 0$ and $1 \le i \le N_{R,\alpha}$ such that $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) = O(1)$. Then, by (4.23), $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) = O(1)$ for all $1 \le i \le N_{R,\alpha}$ and all $R > 0$. Noting that the two equations in (4.6) are satisfied by $x_\alpha = x_{i,\alpha}$ and $\rho_\alpha = \frac{1}{8} d_\alpha$, it follows from (4.8) that the sequence $(\tilde{u}_\alpha)_\alpha$ is uniformly bounded in the balls $B_{\tilde{x}_{i,\alpha}}(1/2)$. Thus, by (4.22) and elliptic theory, the sequence $(\tilde{u}_\alpha)_\alpha$ is bounded in $C^1_{loc}(\mathbb{R}^3)$. Up to a subsequence, still thanks to (4.22), we get that the $\tilde{u}_\alpha$'s converge in $C^1_{loc}(\mathbb{R}^3)$ as $\alpha \to +\infty$ to some $\tilde{u}$ which satisfies

$$\Delta \tilde{u} = \frac{K}{4} \tilde{u}^5$$

in $\mathbb{R}^3$. Moreover, $\tilde{u}$ has two critical points which are 0 and the limit $\tilde{x}_2 \in S^2$ as $\alpha \to +\infty$ of the $\tilde{x}_{2,\alpha}$'s in (4.24). By the classification result of Caffarelli, Gidas, and Spruck [4], this is impossible. In particular, we are left with the second case of our study, where we assume that there exist $R > 0$ and $1 \le i \le N_{R,\alpha}$ such that $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \to +\infty$ as $\alpha \to +\infty$. Then, by (4.23), $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \to +\infty$ as $\alpha \to +\infty$ for all $1 \le i \le N_{R,\alpha}$ and all $R > 0$. The assumptions (4.6) and (4.7) are satisfied by $x_\alpha = x_{1,\alpha}$ and $\rho_\alpha = \frac{1}{8} d_\alpha$. Let $\tilde{v}_\alpha = \tilde{u}_\alpha(0)\tilde{u}_\alpha$. By Lemma 4.1 and (4.22),

$$\Delta_{\tilde{g}_\alpha} \tilde{v}_\alpha + d_\alpha^2 h_\alpha \tilde{v}_\alpha = \frac{F_\alpha}{\tilde{u}_\alpha(0)^4} \tilde{v}_\alpha^5 \ , \tag{4.25}$$

where $\|F_\alpha\|_{L^\infty} \le C$. Noting that $\tilde{u}_\alpha(0) \to +\infty$ as $\alpha \to +\infty$, mimicking again arguments from Druet and Hebey [8], and Druet, Hebey and Vétois [10], we get with (4.25) that, up to a subsequence, $\tilde{u}_\alpha(0)\tilde{u}_\alpha \to \tilde{G}$ in $C^1_{loc}(\mathbb{R}^3 \backslash \{\tilde{x}_i\}_{i \in I})$ as $\alpha \to +\infty$, where the $\tilde{x}_i$'s are the limits of the $\tilde{x}_{i,\alpha}$'s in (4.24), where the index set I is given by $I = \{1, \dots, \lim_{R \to +\infty} \lim_{\alpha \to +\infty} N_{R,\alpha}\}$, and where, for any $R > 0$,

$$\begin{aligned} \tilde{G}(x) &= \sum_{i=1}^{\tilde{N}_R} \frac{\Lambda_i}{|x - \tilde{x}_i|} + \tilde{H}_R(x) \\ &= \frac{\Lambda_1}{|x|} + \left(\sum_{i=2}^{\tilde{N}_R} \frac{\Lambda_i}{|x - \tilde{x}_i|} + \tilde{H}_R(x) \right) \end{aligned} \tag{4.26}$$

in $B_0(R)$, where $\Lambda_i > 0$ for all i, $\tilde{H}_R$ is harmonic in $B_0(R)$, $2 \le \tilde{N}_R \le N_{2R}$ is such that $|\hat{x}_{\tilde{N}_R}| \le R$ and $|\hat{x}_{\tilde{N}_R+1}| > R$, and $N_{2R,\alpha} \to N_{2R}$ as $\alpha \to +\infty$. By Lemma 4.2, and (4.26), we get that

$$\sum_{i=2}^{\tilde{N}_R} \frac{\Lambda_i}{|\tilde{x}_i|} + \tilde{H}_R(0) = 0 \ . \tag{4.27}$$

Independently, by the maximum principle, since $\tilde{G} \ge 0$ and $|\tilde{x}_2| = 1$, there holds that

$$\sum_{i=2}^{\tilde{N}_R} \frac{\Lambda_i}{|\tilde{x}_i|} + \tilde{H}_R(0) \ge \Lambda_2 - \frac{\Lambda_1}{R} - \frac{\Lambda_2}{R-1} \ . \tag{4.28}$$

Choosing $R \gg 1$ sufficiently large, we get a contradiction by combining (4.27) and (4.28). In particular, this proves that $d_\alpha \not\to 0$ as $\alpha \to +\infty$ and thus that there exists $d > 0$, small, such that

$$d_g\left(x_{i,\alpha}, x_{j,\alpha}\right) \ge d$$

for all $1 \le i < j \le N_\alpha$ and all α. Then $\left(N_\alpha\right)_\alpha$ is bounded and it can be assumed to be constant equal to N after passing to a subsequence. Let x_α be a point where u_α is maximum. By (4.3), $u_\alpha(x_\alpha) \to +\infty$ as $\alpha \to +\infty$. Up to passing to another subsequence, we then get that there exists $i \in \{1, \dots, N\}$ such that $d_g\left(x_{i,\alpha}, x_\alpha\right) \to 0$ as $\alpha \to +\infty$ and we get that (4.6) holds true with $\rho_\alpha = \frac{1}{16}d$. Also we have that (4.7) holds true. Hence, by Lemma 4.2, $\rho_\alpha \to 0$ as $\alpha \to +\infty$ and we get a contradiction with $d > 0$. The theorem is proved. $\square$

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